Problems in Finite Model Theory

Last updated August 21, 2003

Bedlewo	2003
Luminy	2000
Oberwolfach	1998
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These pages contain a compilation of open problems in finite model theory, and, when solved, their solutions. The most recent version can be obtained on the Finite Model Theory homepage: www-mgi.informatik.rwth-aachen.de/FMT.

Subscriptions, new problems and announcements of solutions can be sent at any time to

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Contributions should be set preferably in LaTeX and not exceed half a page (problems) or a page (solutions). Any other updates are also welcome, e.g., references, when the announced solutions appear in print.

Contents

1 Bedlewo 2003

1.1 Fixed-point logic with counting — *Andreas Blass*

Can either of the following be expressed in fixed-point logic plus counting:

1. Given a graph, does it have a perfect matching, i.e., a set M of edges such that every vertex is incident to exactly one edge from M?

2. Given a square matrix over a finite field (regarded as a structure in the natural way, as described in [1]), what is its determinant?

It is known that (1) is expressible if restricted to bipartite graphs and that (2) is expressible if the field has only two elements. Both these results are in [1].

Reference

[1] A. Blass, Y. Gurevich, and S. Shelah, *On polynomial time computation over unordered structures*, J. Symbolic Logic 67 (2002) 1093–1125.

1.2 Monadic second-order transductions — *Bruno Courcelle*

For definitions, see www.labri.fr/˜courcell/ActSci.html.

1. Conjecture (Seese): If C is a class of finite graphs for which the satisfiability problem for monadic second-order logic (MS logic) is decidable, then this class is the image of a set T of finite trees under a monadic second-order transduction, equivalently C has bounded clique-width.

Reference: D. Seese, *The structure of the models of decidable monadic theories of graphs*, Ann. Pure Appl. Logic 53 (1991) 169-195.

Observation. For proving Seese's Conjecture, one need to find a monadic second-order compatible graph transformation S and a function f such that, for every graph G, if G has clique-width more than $f(k)$, then $S(G)$ is a $k \times k$ square grid.

2. A stronger conjecture: If C and T are as above, then T is also the image of C under a monadic second-order transduction. Hence a decision algorithm for the satisfiability problem for MS logic in graphs of C can be derived from one for T and vice-versa.

3. Reductions between specific cases.

Let C and D be two classes of graphs. We write $C \geq D$, if one can prove the conjecture for all subsets of D assuming it is proved for all subsets of C.

Do we have "finite graphs" $\geq$ "countable graphs" (the conjecture is actually stated by D. Seese for finite and infinite graphs)?

Do we have "finite graphs" $\geq$ "finite relational structures with relations of arity more than 2"?

4. Which structure transformations are monadic second-order compatible (MS compatible)?

The following transformations are known to be MS compatible: MS transductions, the Shelah-Stupp-Muchnik tree expansion. Compositions of MS compatible transformations are MS compatible. (Unfolding is a composition of an MS-transduction and a tree-expansion.)

Question. Can one find MS-compatible transformations that cannot be obtained as finite compositions of transformations of the above two basic forms.

5. For a Noetherian and confluent term rewriting system, the normal form mapping goes from finite terms to finite terms.

Question. When is it an MS-transduction? When is it MS-compatible?

1.3 Monadic second-order logic with cardinality predicates — *Bruno Courcelle*

The problem concerns the extension of Monadic Second Order Logic (over a binary relation representing the edge relation) with the following atomic formulas:

"Card(X) = Card(Y)"

"Card(X) belongs to A"

where A is a fixed recursive set of integers.

Let us fix k and a closed formula F in this language. Is it true that the validity of F for a graph G of tree-width at most k can be tested in polynomial time in the size of G?

Remark. With the equicardinality predicate, MS logic becomes undecidable on words because satisfiability of Post Correspondence Problem can be encoded.

1.4 ESO arity hierarchy — *Etienne Grandjean*

For any first-order signature σ, let $\textsc{Ntime}^{\sigma}(n)$ denote the class of σ-problems (= sets of σ-structures) that are recognized by nondeterministic RAMs in time $O(n)$ where n is the cardinality of the domain of the input σ-structure. Let $\mathrm{ESO}^{\sigma}(\forall 1)$ (resp. $\mathrm{ESO}^{\sigma}(\mathrm{arity}\, 1)$) denote the class of σ-problems defined by ESO (Existential Second Order) formulas with only one first-order variable (resp. with ESO function and relation symbols of arity non greater than 1). The class $\mathrm{ESO}^{\sigma}(\mathrm{arity}\, 1, \forall 1)$ is defined similarly.

The following equalities are proved in [1] for any signature σ:

$$\textsc{Ntime}^{\sigma}(n) = \mathrm{ESO}^{\sigma}(\forall 1) = \mathrm{ESO}^{\sigma}(\mathrm{arity}\, 1, \forall 1).$$

(Note: This class is denoted vertexNLIN$^{\sigma}$.)

Conjecture. For any signature σ of arity 1, the following equality also holds:

$$\mathrm{ESO}^{\sigma}(\forall 1) = \mathrm{ESO}^{\sigma}(\mathrm{arity}\, 1).$$

Remark. By Cook's theorem on time hierarchy and by Fagin's padding technics in FMT, that conjecture would imply that the following "Fagin's arity hierarchy" is strict at each level: For any signature σ of arity 1 and any d greater than or equal to 1, $\mathrm{ESO}^{\sigma}(\mathrm{arity}\, d)$ is strictly included in $\mathrm{ESO}^{\sigma}(\mathrm{arity}\, d+1)$.

Reference

[1] E. Grandjean and F. Olive, "Graph properties checkable in linear time in the number of vertices" (56 pages), to appear in JCSS

1.5 A gap in the complexity of natural problems — *Etienne Grandjean*

All the natural graph or digraph problems (given by their adjacency nxn matrix, i.e. as $\{E\}$-structures where E is a binary relation) that are presented in [1] are either in $\textsc{Ntime}^{\{E\}}(n)$ (= $\mathrm{ESO}^{\sigma}(\forall 1)$) or in $\textsc{Ntime}^{\{E\}}(\Omega(n^2))$.

Question. Exhibit a natural graph or digraph problem that is neither in $\textsc{Ntime}^{\{E\}}(n)$ (= $\mathrm{ESO}^{\sigma}(\forall 1)$) nor in $\textsc{Ntime}^{\{E\}}(\Omega(n^2))$, that means: is recognized (by some nondeterministic RAM) in time $o(n^2)$ but not in time $O(n)$.

Remark. The "nonnatural" set of graphs that have at least $n^{3/2}$ edges (where n is the number of vertices) obviously belongs to $\textsc{Ntime}^{\{E\}}(n^{3/2})$.

Reference

[1] E. Grandjean and F. Olive, "Graph properties checkable in linear time in the number of vertices" (56 pages), to appear in JCSS

1.6 Three open problems concerning dynamic complexity — *Neil Immerman*

1. Are REACH_u or REACH(*acyclic*) in DYNQFREE?
2. REACH is not complete via bfop's for DYNFO(COUNT); but is it complete for this class via slightly stronger reductions?
3. Is REACH restricted to graphs of bounded tree width in DYNFO?

References

William Hesse and Neil Immerman: *Complete Problems for Dynamic Complexity Classes.* LICS 2002: 313 –

A positive answer to (1) – Bill Hesse

Bill Hesse has proved that REACH_u — undirected graph reachability — is in fact in DYNQFREE.

This result along with much related work can be found in his Ph.D. thesis a draft of which is already on-line: http://www.cs.umass.edu/˜whesse/thesis.ps.

1.7 Frail 0-1 laws — *Jean-Marie Le Bars*

0-1 laws are usually obtained by considering the uniform distribution. Is it possible to change the issue by a slight modification of the distribution?

Let $\mathcal{R}$ be a relational vocabulary. For each natural number n, we denote by $\mathcal{M}(n, \mathcal{R})$ the set of $\mathcal{R}$-structures with domain $n = \{0, \ldots, n-1\}$. Let a constant $0 < p < 1$, the following experiment yields a random structure M_n of $\mathcal{M}(n, \mathcal{R})$ over the probability space $E(n,p)$: for any natural number k, any $S \in \mathcal{R}$ of arity k and any tuple $\overline{a_k}$ of k elements of n, we choose at random with probability p whether the tuple $\overline{a_k}$ is in the relation S. For any property $\mathcal{P}$ on $\mathcal{R}$-structures, we denote by $\mu_{n,p}(\mathcal{P})$ the probability that $\mathcal{P}$ holds on M_n. Let $\mathcal{L}$ a logic, a 0-1 law holds for $\mathcal{L}$ over $E(n,p)$ when, for each property $\mathcal{P}$ expressible in $\mathcal{L}$, the limit of $\mu_{n,p}(\mathcal{P})$ is 0 or 1, when $n \to +\infty$.

Does there exist a (natural) logic $\mathcal{L}$ such that

- the 0-1 law holds for the uniform distribution ($p = 1/2$).
- there exists $\varepsilon > 0$ such that the 0-1 law fails over $E(n,p)$, where $p = 1/2 + \varepsilon$ or $p = 1/2 - \varepsilon$.

1.8 Random graphs with specified degree sequence — *Jim Lynch*

The degree sequence of an n-vertex graph is $d_0, ..., d_{n-1}$, where each d_i is the number of vertices of degree i in the graph. A random graph with degree sequence $d_0, ..., d_{n-1}$ is a randomly selected member of the set of graphs on $\{0, ..., n-1\}$ with that degree sequence, all choices being equally likely. Let $\lambda_0, \lambda_1, \ldots$ be a sequence of nonnegative reals summing to 1. A class of finite graphs has degree sequences approximated by $\lambda_0, \lambda_1, \ldots$ if, for every i and n, the members of the class of size n have $\lambda_i n + o(n)$ vertices of degree i. There is a convergence law for random graphs with degree sequences approximated by some sequence $\lambda_0, \lambda_1, \ldots$. If $\sum_{i=1}^{\infty} i\lambda_i$ is finite and certain other conditions on the sequence $\lambda_0, \lambda_1, \ldots$ hold, then the probability of any first-order sentence on random graphs of size n converges to a limit as n grows.

Some open problems suggested by this convergence law are:

1. What are the possible values that the probability can converge to ?
2. What is the complexity of computing this limit?
3. If $\sum_{i=1}^{\infty} i\lambda_i = \infty$, does the convergence law still hold?
4. Does the convergence law hold for any logical languages more powerful than first-order logic?
5. Of particular interest to internet applications, physics, and biology, is there a general characterization of random graph processes that result in power law distributions?

Reference

James F. Lynch, *Convergence Law for Random Graphs with Specified Degree Sequence*, LICS 2003.

1.9 Arity of LFP-queries — *Greg McColm*

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One of the characterizations of Least Fixed Point logic over positive elementary formulas (LFP(FO), see [7]) consists of taking an *operative system of positive elementary formulas*,

$$\varphi_0(\mathbf{x}_0, \mathbf{S}), \ldots, \varphi_\nu(\mathbf{x}_\nu, \mathbf{S}),$$

where:

- for each i, $\mathbf{x}_i = x_{i,1}, \ldots, x_{i,d_i}$ is a tuple of first order variables,
- the tuple $\bar{S} = S_0, \ldots, S_\nu$ is a tuple of second order variables, where for each i, S_i ranges over d_i-ary relations, and
- for each i, the formula φ_i is positive in $S_0, \ldots, S_\nu$, i.e., there are no $\neg$ symbols "in front of" any term $S_i(\bar{u})$.

The *arity*, or *dimension*, or *number of recursion variables* of this system is $\max\{d_i\colon i = 0, 1, \ldots, \nu\}$. (A game-theoretic version can be found in [5].)

In [1], there appears the question: suppose that the *arity* of a LFP(FO) query is the least arity of any operative system $\bar{\varphi}$ that has that query in its least fixed point. Do the arities of the LFP(FO) expressible queries generate an infinite hierarchy?

This question was answered affirmatively by [2], which exhibited a a sequence of queries such that for each d, there was a boolean-valued LFP(FO)-expressible query R of minimal arity greater than d. Grohe then asked if we restricted our attention to the class of structures with successor relations, does the hierarchy still fail to collapse? In fact, he proved that if the arity hierarchy does not collapse over the class of structures with successor relations, then DLOGSPACE $\neq$ PTIME.

In [6], I generalized his question. Suppose that we have a class of structures which admit a LFP(FO) expressible query that is not FO expressible. Is it true that for any such class of structures, the arity hierarchy does not collapse? I then proved that this conjecture was true when all quantification was "weakly guarded" on a class of structures with a "sparse and uniformly connected" guard relation: the proof used a diagonalization technique.

It should be noted that there is a difference between the arity hierarchy and the Number of Variables hierarchy. In [4], it is proven that over any class of structures admitting an unbounded LFP induction, the Number of Variables hierarchy does not collapse (this proof also uses diagonalization). However, in [3], it is proven that there is a class of structures admitting unbounded LFP inductions, but on which all LFP(FO) queries are FO expressible, and thus of arity 0, so that the arity hierarchy does collapse.

References

[1] A. Chandra & D. Harel, *Structure and complexity of relational queries* **J. Comp. Sys. Sci. 25** (1982) 99–128.

[2] M. Grohe, *Arity hierarchies*, **Ann. Pure and Applied Logic 82** (1996), 103–163.

[3] Y. Gurevich, N. Immerman, S. Shelah, *McColm's Conjecture*, **Proc. 9th IEEE Symp. Logic in Comp. Sci.** (LICS'1994), 10–19.

[4] G. McColm, *Parametrization over inductive relations of a bounded number of variables*, **Ann. Pure and Applied Logic 48** (1990), 103–134.

[5] G. McColm, *Dimension Versus Number of Variables, and Connectivity, too*, **Math. Log. Quart. 41** (1995), 111–134

[6] G. McColm, *The Arity of Least Fixed Point Queries in Guarded Quantification Logic*, in preparation.

[7] Y. Moschovakis, **Elementary Induction over Abstract Structures** (North-Holland, 1974).

1.10 Generic queries — *Luc Segoufin*

Let G be a finite colored graph. Let $<$ be a linear order on G. G is a string if it is a tree with only one branch.

A property is said to be *generic* if it is independent of the linear order. Consider FOg($<$), the generic first-order definable properties over the signature of colored graphs. Gurevich showed that, over finite graphs, FOg($<$) is strictly more expressive than FO.

Question. Does FOg($<$) = FO over finite strings? What about finite trees?

1.11 Locally generic queries — *Michael Abram Taitslin*

Conjecture 1. If, in an expansion of $(\mathbb{N}, <, +)$, locally generic extended queries express more than restricted ones, the first order theory of the expanded domain is undecidable.

Conjecture 2. If, in an expansion of $(\mathbb{N}, <, +)$, locally generic extended queries express more than restricted ones, then there is a first-order formula $\Phi(x, y)$ in the language of the expansion such that for any finite subset A of natural numbers, there is a natural number

b such that $A = \{a \in \mathbb{N} \mid \Phi(a,b)\}$. In particular, the Random Graph is interpretable in the expansion.

Conjecture 3. Does the collapse theorem hold for $< \mathbb{N}, <, +, |_p >$?

1.12 Bracketing-invariant properties — *Jan Van den Bussche*

Consider strings over a finite alphabet Σ, represented in the well-known way as finite structures over $(<, \Sigma)$, where $<$ is a total order and each letter in Σ serves as a unary relation. Expand such strings with a "bracketing:" this is a bijection B from one half of the elements to the other half, such that the situation $x < y < B(x) < B(y)$ never occurs. Only even-length strings have bracketings.

Call a first-order sentence over $(<, \Sigma, B)$ invariant if it does not distinguish between two different bracketings of the same string.

Question. Is every invariant first-order sentence over $(<, \Sigma, B)$ equivalent, on even-length strings, to a first-order sentence just over $(<, \Sigma)$?

Solved – Christof Löding and Thomas Wilke

A positive solution to this problem has in the meantime been announced by Christof Löding and Thomas Wilke. When a preprint of the proof becomes available, a reference will be added to this source.

1.13 A partial (negative) answer to Specker's problem — *Janos A. Makowsky, Eldar Fischer*

E. Fischer has solved the open problem concerning the Specker-Blatter Theorem, which I had contributed some 3-4 years ago (Problem 2.5); more precisely, he has given a counter example, to the extension of the Specker-Blatter Theorem to relation symbols of arity $\geq$ 4.. The solution is posted on my preprint page. I also have a joint paper with E. Fischer on further developments concerning the same material.

The case of arity 3 remains open.

1.14 New results concerning Ash's Conjecture — *Malika More, Annie Chateau*

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Reference

Christopher J. Ash. A conjecture concerning the spectrum of a sentence. *Mathematical Logic Quarterly*, 40:393-397, 1994.

See also the related Problem 3.2.

Let σ denote a relational signature containing the identity relation and $k \geq 2$ a quantifier depth for first-order σ-sentences. For all $n \in \mathbb{N}$, let $N_{\sigma,k}(n)$ be the number of non k-equivalent σ-structures of size n.

Conjecture 1. $\forall\sigma$, $\forall k$, the Ash function $n \mapsto N_{\sigma,k}(n)$ is eventually constant. (Ash 1994)

Conjecture 2. $\forall\sigma$, $\forall k$, the Ash function $n \mapsto N_{\sigma,k}(n)$ is eventually periodic. (Ash 1994)

Theorem 3. If σ is *unary*, then $\forall k$, $N_{\sigma,k}$ is eventually constant. (Ash 1994)

Theorem 4. If $k = 2$, then $\forall\sigma$, $N_{\sigma,2}$ is eventually constant.

Let Φ be a first-order sentence. The first-order spectrum of Φ, denoted by $Sp(\Phi)$, is the set of cardinalities of finite models of Φ. Let NE be the class of binary languages accepted in time $O(2^{cn})$ by a non-deterministic Turing machine, where c is a constant and n is the size of the input. Jones and Selman (1974) proved that NE is the class of all first-order spectra.

Conjecture 5. $\forall\Phi$, $\exists\Psi$, $\mathbb{N} \setminus Sp(\Phi) = Sp(\Psi)$ (Asser 1955)

Thus Asser conjecture is the FMT version of $NE =?$ coNE.

Theorem 6. If (the periodic) Ash conjecture holds, then Asser conjecture holds. (Ash 1994)

For all $i \in \mathbb{N}$, let $F_i = \{n \;/\; N_{\sigma,k}(n) = i\}$. Note that only a finite number of F_i's are nonempty.

Conjecture 7 (very weak Ash conjecture). $\forall\sigma$, $\forall k$, $\forall i$, $F_i \in NE$

Theorem 8. The very weak Ash conjecture holds iff Asser conjecture holds. Moreover, if σ only contains one binary relation and $\forall k$, $\forall i$, $F_i \in NE$, then Asser conjecture holds.

Let $\mathcal{T}$ be a σ-theory. Let $N_{\mathcal{T},k}(n)$ denotes the number of non k-equivalent models of $\mathcal{T}$ of size n. More precisely, let us consider the following theories :

• $\sigma = \{E, =\}$

$\mathcal{T}_b =$ "the binary relation E is bijective"

$\mathcal{T}_f =$ "the binary relation E is functional"

$\mathcal{T}_2 =$ "the binary relation E has total degree less than or equal to 2"

$\mathcal{T}_{2u} =$ "the binary relation E is symmetric, irreflexive and has degree less than or equal to 2"

$\mathcal{T}_{\equiv} =$ "the binary relation E is an equivalence relation"

• $\sigma = \{E_1, E_2, =\}$

$\mathcal{T}_{\subseteq} =$ "the binary relations E_1 and E_2 are equivalence relations and E_1-classes are subsets of E_2-classes"

Theorem 9. For all $k \geq 2$, the Ash functions $N_{\mathcal{T}_b,k}$, $N_{\mathcal{T}_f,k}$, $N_{\mathcal{T}_2,k}$, $N_{\mathcal{T}_{2u},k}$, $N_{\mathcal{T}_{\equiv},k}$, $N_{\mathcal{T}_{\subseteq},k}$ are eventually periodic. (Chateau 2003)

Theorem 10. For all $k \geq 3$, the Ash functions $N_{\mathcal{T}_b,k}$ are *not* eventually constant. (Chateau 2003)

2 Luminy 2000

2.1 Successor- vs. order-invariance — *Heinz-Dieter Ebbinghaus*

Submitted by H. D. Ebbinghaus who attributes the problem to S. Abiteboul, R. Hull and V. Vianu; A. Dawar.

In the following, sentence means first-order sentence.

A sentence φ of vocabulary $\tau_{succ} := \tau \cup \{succ\}$ is called *successor-invariant (in the finite)* if for every (finite) τ-structure $\mathbf{A}$ and successor-relations $succ_1$, $succ_2$ on A,

$$(\mathbf{A}, succ_1) \models \varphi \text{ iff } (\mathbf{A}, succ_2) \models \varphi.$$

The definition of order-invariance (in the finite) is similar. By the interpolation theorem, order-invariant sentences of vocabulary $\tau_<$ and successor-invariant sentences of vocabulary τ_{succ} are equivalent to τ-sentences. On the other hand, there are $\tau_<$-sentences that are order-invariant in the finite, but in the finite are not equivalent to a sentence of vocabulary τ.

Question. Is any τ_{succ}-sentence that is successor-invariant in the finite, equivalent in the finite to a τ-sentence?

2.2 Strategies for EF-games — *Ron Fagin*

Find a winning strategy for the duplicator for some **EF** game that involves more than one coloring round, and use it to prove an inexpressibility result. Two examples of formulas whose **EF** game involves more than one coloring round are:

1. $\exists A_1 \forall x \exists A_2 \phi$, where A_1 and A_2 are unary second-order (set) variables, where x is a first-order (individual) variable, and where ϕ is a first-order formula. Such a formula represents some class in NP (in fact, in closed monadic NP).
2. $\exists A_1 \forall A_2 \phi$, where A_1 and A_2 are unary second-order (set) variables, and where ϕ is a first-order formula. Such a formula represents some class in the second level of the monadic hierarchy.

2.3 Circular order — *Bruno Courcelle*

Mail: courcell@labri.fr, URL: www.labri.fr/Perso/˜courcell/ActSci.html.

Let D be a finite set of size at least 3. A *circular order* on D is a ternary relation R such that, there exists on D a strict linear order $<$ such that:

$$Rxyz \text{ iff } x < y < z \text{ or } y < z < x \text{ or } z < x < y.$$

A circular order satisfies the following properties for all $x, y, z, t \in D$:

A1: $Rxyz$ implies $Ryzx$;

A2: $Rxyz$ and $Rytz$ imply $Rxyt$ and $Rxtz$;

A3: $Rxyz$ implies that $Rxzy$ does not hold;

A4: either $Rxyz$ or $Rxzy$ or $x = y$ or $y = z$ or $x = z$.

Let C be a ternary relation on D. It is *consistent*, if it is included in some circular order R.

Deciding whether C is consistent is **NP**-complete by Galil et Megiddo (Theoretical Computer Science 5 (1977) 179-182).

Question 1. Is the consistency of C expressible by a formula of monadic second-order logic (with C the only relation and without any other constant or relation symbol)?

(This is true if C contains the union of two circular orders on two subsets of D the union of which is D.)

If C is consistent, its **closure** $f(C)$ is defined as the intersection of all circular orders containing C; we let C^* denote its *transitive closure* i.e., the least set of triples satisfying A1 and A2. Using the fact that the transitive closure can be computed in polynomial time, and unless $\mathbf{P} = \mathbf{NP}$, $C^* = f(C)$ does not hold in general.

Question 2. Exhibit a smallest consistent set C such that C^* differs from $f(C)$.

Unless $\mathbf{P} = \mathbf{NP}$, $f(C)$ is not the least fixed point of any monotone operator on sets of triples over D having a polynomial complexity.

We say that C is *strongly consistent* if it is consistent and $f(C)$ is a circular order, which means that C generates a unique circular order.

Question 3. Characterize the strongly consistent sets C.

2.4 Fine spectrum — *John Baldwin*

Submitted by John Baldwin, University of Illinois at Chicago, www.math.uic/˜jbaldwin.

Solve the fine spectrum problem for L^k theories.

That is, analogously to Shelah's work for arbitrary stable theories find all functions from ω to ω such that there is complete L^k theory which has $f(n)$ models with cardinality n.

Work of Djordjevic shows under suitable amalgamation hypotheses every such theory is either unstable or ω-stable. This removes many of the cases in the infinite analysis. But such interesting cases as multidimensionality remain. There should also be a close connection with the theory of homegeneous models as developed by Cherlin, Hrushovski and Lachlan. (See Cherlin's webpage: http://www.math.rutgers.edu/˜cherlin/).

Note that this problem and the following problem submitted by Makowsky differ essentially in counting labeled versus unlabeled structures.

2.5 Specker's Problem — *Janos A. Makowski*

Contributed[1] by J.A. Makowsky.

Counting Labeled Structures mod m

Let C be a class of finite structures for one binary relation symbol R. We define for $A = \{1, \ldots, n\}$

$$F_C(n) = \mid \{R^A \subseteq A^2 : \langle A, R^A \rangle \in C\} \mid$$

Examples:

1. If $C = U$ consists of all R-structures, $f_U(n) = 2^{n^2}$.
2. If $C = B$ consists of bijections, $f_B(n) = n!$
3. If $C = G$ is the class of all (undirected, simple) graphs, $f_G(n) = 2^{\binom{n}{2}}$.
4. If $C = E$ is the class of all equivalence relations, $f_E(n) = B_n$, the *Bell Numbers*.
5. If $C = E_2$ is the class of all equivalence relations with two classes only, of the same size, $f_{E_2}(2n) = \frac{1}{2} \cdot \binom{2n}{n}$. Clearly, $f_{E_2}(2n+1) = 0$.
6. If $C = T$ is the class of all trees, $f_T(n) = n^{n-2}$, *Caley*.

[1]Our exposition follows closely [Blatter and Specker, 1984]

We observe the following:

$$f_C(n) = 2^{n^2} = (-1)^{n^2} \pmod 3$$

$$f_C(n) = n! = 0 \pmod m \text{ for } n \geq m$$

And for each m the functions, $f_G(n) = 2^{\binom{n}{2}}$, $f_E(n) = B_n$, $f_T(n) = n^{n-2}$ are ultimately periodic $\pmod m$.

However, $f_{E_2}(2n) = \frac{1}{2} \cdot \binom{2n}{n} = 1 \pmod 2$ iff $n = 2^{2k}$, hence is not periodic $\pmod 2$.

Enter Monadic Second Order Logic

The first four examples (all relations, all bijections, all graphs, all equivalence relations) are definable in First Order Logic FOL. The trees are definable in Monadic Second Order Logic $MSOL$..

E_2 is definable in Second Order Logic SOL, but not $MSOL$-definable. If we expand E_2 to have the bijection between the classes we get structures with two binary relations. The class is now FOL-definable. Let us denote the corresponding counting function $F_{E_2}(2n)$. We have

$$f_{E_2}(2n) \cdot n! = F_{E_2}(n) = 0 \pmod m$$

for n large enough.

Periodicity and Linear Recurrence Relations

The periodicity of $f_C(n) \pmod m$ is usually established by exhibiting a *Linear Recurrence Relation*:

There exists $1 \leq k \in \mathbb{N}$ and integers $a_1, \ldots, a_k$ such that for all n

$$f_C(n) = \sum_{j=1}^{k} a_j \cdot f_C(n-j) \pmod m$$

Examples:

1. In the case of $f_C(n) = 2^{n^2}$ we have

$$f_C(n) = f_C(n-2) + 2 \cdot f_C(n-1) \pmod 3$$

2. In the case of $f_C(n) = n!$ we have for all m

$$f_C(n) = 0 \cdot f_C(n-1) \pmod m$$

In this case we say that f_C *trivializes*.

Blatter-Specker Theorem

Theorem: Let τ be a binary vocabulary, i.e. all relation symbols are at most binary. If C is a class of finite τ-structures which is $MSOL$-definable, then for all $m \in \mathbb{N}$ $f_C(n)$ is ultimately periodic $\pmod{m}$.

Moreover, there exists $1 \leq k \in \mathbb{N}$ and integers $a_1, \ldots, a_k$ such that for all n

$$f_C(n) = \sum_{j=1}^{k} a_j \cdot f_C(n-j) \pmod{m}$$

i.e we have a linear recurrence relation.

Problem: Does this hold also for **ternary (arbitrary finitary)** relation symbols?

References

- C. Blatter and E. Specker, *Recurrence relations for the number of labeled structures on a finite set.* In Logic and Machines: Decision Problems and Complexity, E. Börger, G. Hasenjaeger and D. Rödding, eds, LNCS 171 (1984) pp. 43-61
- E. Specker, *Application of Logic and Combinatorics to Enumeration Problems*, In: Trends in Theoretical Computer Science, E. Bẅger ed., Computer Science Press, 1988, pp. 141-169 Reprinted in: Ernst Specker, Selecta, Birkhäuser 1990, pp. 324-350.

A partial (negative) answer – Janos A. Makowsky, Eldar Fischer

Communicated by Janos A. Makowsky

This problem was solved by E. Fischer; more precisely, he has given a counter example, to the extension of the Specker-Blatter Theorem to relation symbols of arity $\geq$4.. The solution is posted on my preprint page. I also have a joint paper with E. Fischer on further developments concerning the same material.

The case of arity 3 remains open.

3 Oberwolfach 1998.

3.1 Homomorphic embeddings — *Moshe Vardi et al.*

Let B be a finite relational structure. Let C_B be the set of finite A that can be homomorphically embedded on A.

Conjecture. C_B is either decidable in polynomial time or NP-complete (depending on the choice of B). (It is sufficient to consider directed graphs.)

Reference

Vardi and Feder: see Vardi's homepage.

3.2 A spectrum conjecture — *Christopher J. Ash*

Fix a finite relational vocabulary τ. Consider the (finite) list of all equivalence classes of finite τ defined by equivalence by formulas of quantifier rank at most k. (This is NOT prefix rank). Let $S_{\tau,k}(n)$ be the number of such classes which are realized by a structure of size n. As a function of n for fixed k and τ is each such function eventually constant? (NP = co-NP implies the functions are not eventually constant.) Try to work out the functions for a single binary function and $k = 3$.

Reference

C.J. Ash, *A conjecture concerning the spectrum of a sentence*, Math Logic Quarterly (1994) 393-397.

4 Luminy 1995

4.1 Three questions — *Scott Weinstein*

The first two problems are posed for $k \geq 3$. Martin Otto has shown that these problems reduce to the case $k = 3$.

1. A finite structure A is *k-small* if and only if no structure of size less than A is L^k-equivalent to A. The set of k-small finite structures is co-NP. Is it co-NP complete?

2. A finite structure A is *k-categorical* if and only if every finite structure that is L^k-equivalent to A is isomorphic to A. The set of k-categorical finite structures is co-r.e. Is it co-r.e. complete?

3. Let EXT be the class of boolean queries that are closed under extensions. Is FO $\cap$ EXT $\subseteq L^{\omega}_{\infty\omega}(\exists)$?

One answer - Martin Grohe

Martin Grohe answered question 3 negatively. A proof, which generalizes an idea Rosen and Weinstein used to prove the failure of existential preservation for $L^{\omega}_{\infty\omega}$, can be found in the paper "Existential least fixed-point logic and its relatives". It is available via www on page www.dcs.ed.ac.uk/home/grohe/.

4.2 Is TC$\cap$Datalog = pos$\exists$TC ? — *Martin Grohe*

Is $\mathbf{TC} \cap \mathbf{Datalog} = \mathbf{pos\exists TC}$?

Remark. It is known that $\mathbf{FO} \cap \mathbf{Datalog} = \mathbf{pos\exists FO}$,
but also that $\mathbf{TC} \cap \mathbf{Datalog(\neg, \neq)} \supset \mathbf{\exists TC}$.

Notation. **TC** denotes transitive closure logic, $\mathbf{\exists TC}$ its existential fragment, and $\mathbf{pos\exists TC}$ its positive existential fragment. The notation is similar for the fragments

pos∃FO and **∃FO** of first–order logic **FO**. **Datalog** is pure Datalog without any negation symbols, and **Datalog(¬, ≠)** the extension where negation symbols are allowed in front of extensional relation symbols and equalities.

4.3 Cofinite laws — *Ron Fagin*

The *spectrum* of a first-order sentence σ is the set of cardinalities of the finite structures that satisfy σ. Let us say that a logic has a *cofinite law* if for every sentence σ of the logic, either σ or $\neg\sigma$ has a cofinite spectrum. In my original 0-1 law paper, I noted that the 0-1 law implies the cofinite law. The same is true for any logic with a limit law (Proof: any sentence with positive asymptotic probability has a cofinite spectrum). There is also an easy direct proof (see my 0-1 law paper). There are logics with a cofinite law but no limit law. For example, Pacholski noted that a logic without equality has a cofinite law, but, as in the case of monadic existential second-order logic without equality, not necessarily a limit law. Tyszkiewicz noted that even logics with equality can have a cofinite law without a limit law (example: $L^{\omega}_{\infty,\omega}$ over a unary function: this has a cofinite law by a similar easy argument to that given in my 0-1 law paper). A cofinite law is capable of giving an inexpressibility result (for example, it implies that EVENNESS is not definable). Can we find an interesting application of a cofinite law in some logic (say, to prove a nontrivial inexpressibility result)?

4.4 Spectra of categorical sentences — *Ron Fagin*

A sentence is *categorical* if it has at most one model of each finite cardinality. Is every spectrum the spectrum of a categorical sentence? This is true if P = NP (or even if UEXP = NEXP). This problem is discussed in my survey paper.

4.5 Hanf and Löwenheim numbers in the finite — *Anuj Dawar*

For the relevant definitions of L^k-types and equivalence, refer to: [A. Dawar, S. Lindell, and S. Weinstein. Infinitary logic and inductive definability over finite structures. Technical Report MS-CIS-91-97, University of Pennsylvania, 1991. Revised version to appear in *Information and Computation*.]

1. Is there a recursive function f_k, for every k, such that, for every finite structure A which realizes n distinct L^k-types, there is a B such that $\text{card}(B) \leq f_k(n)$ and $A \equiv^k B$?

2. Is there a recursive function g_k for every k, such that, if A is a finite structure realizing n distinct L^k-types, and $\text{card}(A) \geq g_k(n)$, then there are arbitrarily large finite B such that $A \equiv^k B$?

Solved – Martin Grohe, Russel Barker

The first question was answered negatively by Martin Grohe, in [2].

The second question was answered negatively by Russel Barker, in [1].

References

[1] Russel Barker: *There is no recursive link between the k-size of a model and its cardinality.* Ann. Pure Appl. Logic 118 (2002), no. 3, 235–247.

[2] Martin Grohe: *Large finite structures with few L^k-types.* Information and Computation 179(2): 250-278, 2002.

4.6 Monadic second-order logic with auxiliary invariant linear order — *Bruno Courcelle*

Let L be a class of finite structures $\langle D, R, < \rangle$, where D is the domain, R a binary relation, $<$ a linear order on D. Assume that for any two orders $<$ and $<'$ $\langle D, R, < \rangle$ belongs to L iff $\langle D, R, <' \rangle$ does. Assume that L is definable by a monadic second-order formula (among all finite structures of same type). Let L' be the corresponding set of structures $\langle D, R \rangle$ (i.e., such that $\langle D, R, < \rangle$ belongs to L for some $<$).

Is it always true that L' is definable by a formula of counting monadic second-order logic, i.e., a formula of monadic second-order logic using also special quantifiers saying that "the number of elements x of D satisfying ..." has is a multiple of some fixed integer n. Conjecture is NO.

See more details in: B. Courcelle, *The monadic second-order logic X: Linear orders*, URL: www.labri.fr/˜courcell/ActSci.html, Mail: courcell@labri.u-bordeaux.fr

4.7 Definability of NP graph properties, revisited — *John Lynch*

In the Oberwolfach Conference on Finite Model Theory, two of the problems that were posed were:

8. *(E. Grandjean)* Investigate the class of graph properties π that can be defined by an existential second-order formula with unary functions only, i.e.

$G = (V, E)$ belongs to π
iff G satisfies a formula of the form $\exists f_1, \dots, f_k\ \psi(E, f_1, \dots, f_k)$
(where the f_i are unary functions).

10. *(J. Lynch)* Let L_k be the fragment of binary second-order language of graphs consisting of sentences of the form

$$\exists R_1 \dots \exists R_k \forall x_1 \dots \forall x_k \exists y_1 \dots \exists y_k\ \tau(x_1, \dots, x_k, y_1, \dots, y_k, E, R_1, \dots, R_k)$$

where $R_1, \dots, R_k$ are binary second-order variables and E is interpreted as the graph edge relation. Let $L = \bigcup_k L_k$.

(a) Find an isomorphism invariant property of graphs that is not definable in L. That is, find a collection of graphs $\mathcal{C}$ such that there is no $\sigma \in L$ for which

$$G \in \mathcal{C} \quad \Longleftrightarrow \quad G \models \sigma\ .$$

(b) Similar, but for L_k. For what k does this become hard?

These problems are still unsolved, and they appear to be difficult. So here are some restricted versions of them. Whether they are easier than the original problems remains to be seen.

(a) S is a successor relation if it is isomorphic to the usual successor relation on a finite segment of the natural numbers. Investigate the class of graph properties π that can be defined by an existential second-order formula with successor relations only, i.e.

$G = (V, E)$ belongs to π
iff G satisfies a formula of the form $\exists S_1, \dots, S_k\ \psi(E, S_1, \dots, S_k)$
(where the S_i are successor relations).

(b) Let L_k be the second-order language of graphs consisting of sentences of the form

$$\exists S_1 \dots \exists S_k \forall x_1 \dots \forall x_k \exists y_1 \dots \exists y_k\ \tau(x_1, \dots, x_k, y_1, \dots, y_k, E, S_1, \dots, S_k)$$

where $S_1, \dots, S_k$ are second-order variables interpreted as successor relations, and E is interpreted as the graph edge relation. Let $L = \bigcup_k L_k$.

Find an isomorphism invariant property of graphs that is not definable in L. Also, for a given k, find a property not definable in L_k.

4.8 About Convergence Laws For Finite Structures
— *John Lynch*

4.8.1 Tree-like structures

1. Is there an extended convergence law for the first-order theory of several random unary functions? An extended convergence law, as defined in j. Lynch, An extension of 0-1 laws, *Random Structures and Algorithms* **5**, 155–172 1994, states that for every sentence of the language in question, its probability is asymptotic to $\beta n^{-\gamma}$ for some $\beta > 0$ and $\gamma \geq 0$, or it is smaller than the reciprocal of any polynomial in n.

2. Is there an extended convergence law for the monadic second-order theory of one random unary function?

3. Are there convergence laws for the first-order theory of several weighted random mappings (as in the thesis of A. Broder, Weighted Random Mappings; Properties and Applications, Stanford University 1985)?

4. Are there convergence laws for the monadic second-order theory of one weighted random mapping?

5. Is there a convergence law for the first-order theory of several random partial functions (as in Jaworski's thesis, or J. Jaworski, On a Random Digraph, *Ann. Disc. Math.* **33**, 111–127, 1987)?

6. Is there a convergence law for the monadic second-order theory of one random partial function?

7. Is there a convergence law for languages recognized by probabilistic tree automata (analogous to Markov chains with variable transitions)?

8. Are there convergence laws for theories of complete trees, or trees with bounded degree?

4.8.2 Graphs

1. Is there a convergence law for the monadic second-order theory of a random regular graph?

2. Is there a strong 0-1 law for graphs, having the same relationship to the 0-1 law that the strong law of large numbers has to the weak law of large numbers (posed by Mycielski)?

3. Is there an extended convergence law for the first-order theory of a random graph with edge probability n^{-1}?

4. Is there a 0-1 law for the first-order theory of pseudo-random graphs (posed by Grädel)?

4.8.3 Other structures, with variable probabilities

1. Are there convergence laws for first-order theories of several relations of degree greater than two (posed by Dolan)?

2. Is there an extended convergence law for first-order theories about random relations with a built-in successor relation (posed by Lynch)?

5 Oberwolfach 1994

5.1 Characterization theorems — *Heinz-Dieter Ebbinghaus*

In classical model theory, the methodological scope of important logics has been clarified by characterization theorems such as Lindström's theorems on first-order logic or Barwise's theorem on $L_{\infty\omega}$. In order to explore model theoretic properties in the finite, one should try to perform a similar program for this case. In particular, is there a Lindström-type theorem for fixed-point logic ?

5.2 Monadic NP quantifier hierarchy — *Ron Fagin*

A *monadic NP* class is a class defined by an existential monadic second-order sentence.

Problem: Does one unary relation symbol suffice ? That is, is every monadic NP class (such as 3-colorability) definable as $\exists S\phi$, where S is a single unary relation symbol?

Solved – M. Otto

This problem was solved by M. Otto: The number of monadic quantifiers in monadic existential second-order logic gives rise to a strict hierarchy in the finite. The proof appeared in *Information Processing Letters* **53**(1995), pp. 337–339.

5.3 Monadic quantifier alternation hierarchy — *Ron Fagin*

The *monadic hierarchy* consists of classes definable by sentences $Q_1S_1 \ldots Q_kS_k\phi$, where the S_i's are unary relation symbols, the Q_i's are $\exists$ or $\forall$, and ϕ is first-order.

Problem: Is the hierarchy strict? Or does it collapse to some fixed number of alternations of second-order quantifiers?

Fact: If the polynomial hierarchy is strict then so is the monadic hierarchy. But we would like to prove this without complexity assumptions.

Solved – Oliver Matz, Wolfgang Thomas, and Nicole Schweikardt

This problem was solved by Oliver Matz, Wolfgang Thomas, and Nicole Schweikardt: Matz and Thomas showed that the monadic hierarchy is strict for the class of finite structures, the class of finite graphs, and the class of coloured grids (LICS 1997, "The monadic quantifier alternation hierarchy over graphs is infinite"). Schweikardt refined their proof and showed that the monadic hierarchy is strict even for the class of uncoloured grids (CSL 1997, "The monadic quantifier alternation hierarchy over grids and pictures"). A joint paper which combines and extends these results, will appear in a LICS 1997 special issue of Information and Computation (Matz, Schweikardt, Thomas, "The monadic quantifier alternation hierarchy over grids and graphs").

5.4 0 – 1 laws for 2-variable logics — *Jörg Flum*

FO^2, the fragment of FO consisting of sentences with at most two variables, is decidable. Does Σ_1^1 over FO^2 have a 0–1-law?

Solved – Jean-Marie le Bars

This question was solved negatively by Jean-Marie le Bars.

Reference

Jean-Marie Le Bars: *Fragments of Existential Second-Order Logic without 0-1 Laws.* LICS 1998: 525-536

5.5 Definability of NP-complete problems — *Erich Grädel and Anuj Dawar*

Question 1. Is PLANARITY in FP or $L^{\omega}_{\infty\omega}$?

Question 2. Is 3-COLOURABILITY in $L^{\omega}_{\infty\omega}$?

Question 3. Similarly for other natural NP-complete problems on unordered structures . . .

Solved – Martin Grohe, Anuj Dawar

The first question was answered positively by Martin Grohe in [2].

The second question was answered negatively by Anuj Dawar in [1].

References

[1] Anuj Dawar: *A Restricted Second-Order Logic for Finite Structures*, Information and Computation, 143 (1998) 154-174.

[2] Martin Grohe: *Fixed-point logics on planar graphs*, in Proceedings of the 13th Annual IEEE Symposium on Logic in Computer Science, 1998.

5.6 Linear reduction between theories — *Etienne Grandjean*

Prove that the decision problem T_2 of the first-order random theory of *two binary relations* is reducible to the similar problem T_1 *for only one binary relation* via a linear time bounded reduction. That would imply that T_1 and T_2 have exactly the same time complexity.

5.7 Complexity of sorting circuits — *Etienne Grandjean*

Show that for each n there is a boolean circuit C_n that can sort n integers (in binary notation) in the range $0, \ldots, n^k$ (k fixed) so that the circuit C_n

- has $O(n(log\ n)^2)$ gates
- and is computable in time $O(n(log\ n)^3)$ (that means: in time linear in the length of its description) on a Turing machine.

Remark: We hope to get this result by a careful inspection of the sorting network of [Ajtai, Komlos, Szemeredi, 1993] which sorts n integers in time $O(log\ n)$ with n registers.

5.8 NP graph properties definable with unary functions
— *Etienne Grandjean*

Investigate the class of graph properties π that can be defined by an existential second-order formula with unary functions only, i.e.

$G = (V, E)$ belongs to π
iff G satisfies a formula of the form $\exists f_1, \ldots, f_k\ \psi(E, f_1, \ldots, f_k)$
(where the f_i are unary functions).

(a) Study the extent and the robustness of this class.

(b) Prove that some problems do not belong to it. (E.g. is G edge-colorable with d colors where $d = \mathit{degree}(G)$?)

5.9 A Preservation Problem in Finite Model Theory
— *Phokion Kolaitis*

Suppose ϕ is a first-order sentence that is preserved under extensions and homomorphisms on finite models. Is ϕ equivalent to an existential positive sentence on finite models ?

Background: (a) preservation under extensions fails finitely (Tait/Gurevich & Shelah)

(b) preservation under homomorphisms fails finitely (Ajtai & Gurvich)

(c) A positive answer to the above problem will yield as an easy corollary the theorem of Ajtai & Gurevich that a Datalog program is bounded if and only if it defines a first-order property.

5.10 NP graph properties definable with binary relations
— *John Lynch*

Let L_k be the fragment of binary second-order language of graphs consisting of sentences of the form

$$\exists R_1 \ldots \exists R_k \forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_k\ \tau(x_1, \ldots, x_k, y_1, \ldots, y_k, E, R_1, \ldots, R_k)$$

where $R_1, \ldots, R_k$ are binary second-order variables and E is interpreted as the graph edge relation. Let $L = \bigcup_k L_k$.

(a) Find an isomorphism invariant property of graphs that is not definable in L. That is, find a collection of graphs $\mathcal{C}$ such that there is no $\sigma \in L$ for which

$$G \in \mathcal{C} \iff G \models \sigma \,.$$

(b) Similar, but for L_k. For what k does this become hard?

5.11 Random structures — *John Lynch*

What classes of random finite structures have relevance to:

(a) database theory;

(b) statistical verification of programs;

(c) algorithm analysis.

5.12 Effective syntax for order-invariance — *Janos A. Makowsky*

Let L be a logic, $\phi \in L[\tau \cup R]$.

Definition: ϕ is *invariant under order* if for every finite τ-structure $\mathbf{A}$ and two linear orderings $<_1^A, <_2^A$ on it we have $(\mathbf{A}, <_1^A) \models \phi \Longleftrightarrow (\mathbf{A}, <_2^A) \models \phi$.

Problem: Is there a logic $L \subset FP$ or $\subset L^{\omega}_{\infty\omega}$ such that for every order invariant ϕ there is $\psi \in L[\tau]$ (without order symbol) such that $\phi \equiv \psi$?

Notes: (a) If L satisfies the Interpolation Theorem (Δ-Interpolation suffices) then this is true.

(b) There is an order-invariant ϕ in FOL such that no such ψ exists (Gurevich).

5.13 Generalized Quantifiers and 0-1 laws — *Iain Stewart*

Take an arbitrary NP-complete problem Ω and incorporate the corresponding sequence of Lindström quantifiers into first-order logic to get the logic $(\pm\Omega)^*[FO]$ (there is a quantifier for each arity à la Immerman's transitive closure logic, but no built-in relations such as successor; also "$\pm$" tells us that we can apply Ω within negation signs and "*" tells us that we may nest applications of Ω as we like).

Question 1. Is there a logic $(\pm\Omega)^*[FO]$ with a 0–1-law ?

Note that HP*[FO] = NP (Dahlhaus) if we have two constants available. (HP = Hamiltonian Path)

Partially solved by Anuj Dawar and Erich Grädel

A solution to the first part is implied by results of A. Dawar and E. Grädel: Generalized Quantifiers and 0-1 Laws, Proc. LICS'95. The paper is available from a.dawar@swansea.ac.uk or graedel@cs.rwth-aachen.de.

Question 2. Does there exist a complete problem for NP via quantifier-free translations without constants or built-in relations ?

This second question harks back to Lovász and Gács (1977). Also, Blass an Harary have asked for a logic L which can express Hamilton Cycle and which has a 0–1–law.

Solved – Henrik Imhof

This problem has been solved by Henrik Imhof:

A modification of the class of satisfiable Boolean circuits yields a problem which is NP-complete with respect to quantifier free reductions. This class had also been studied by Lovasz and Gacs, who considered slightly different reductions. A completeness proof and a presentation of circuit quantifiers for a variety of logics (including the case NP) can be found in "Fixed Pont Logics, Generalized Quantifiers, and Oracles." This paper is available at www.dcs.ed.ac.uk/home/grohe/pub.html.

5.14 0-1 laws and collaps to FO: I — *Jurek Tyszkiewicz*

Is the following conjecture true:

If for a recursive distribution $\boldsymbol{\mu}$ on the class of all finite models of a signature τ 0-1 laws hold both for MSO and FP, then for every formula $\phi(\vec{x})$ in MSO $\cup$ FP there is a formula $\psi(\vec{x})$ in FO such that $\mu(\forall\vec{x}\,\phi \leftrightarrow \psi) = 1$.

This problem was solved by Monica McArthur mcarthur@math.ucla.edu.

Answer: No. We give a sketch of the counterexample.

Let T_k be the conjunction of all extension axioms (in the language with one binary relation) with $\leq k$ variables. Let $b(k)$ be a recursive function such that for all $m \geq b(k)$, there is a model of T_k of cardinality m. Now, we define a sequence $\{A_i\}$ of connected directed graphs, $|A_i| = i$; each A_i, $i \geq b(3)$ has the essential property that it satisfies T_j for all j such that $b(j) \leq i$. It is clear that the set $A = \{A_i\}$ is recursive. A has an $L^{\omega}_{\infty,\omega}$ 0-1 law but not an MSO 0-1 law; the lack of an MSO 0-1 law is due to a theorem of Tyszkiewicz that a class with a recursive measure on which the extension axioms have probability 1 does not have an MSO convergence law.

Now, we define a class A^* in such a way that A^* has exactly one member, B_n, of size n for each n, and for each k there is an n such that each structure of A^* of size greater than n has more than k copies of each A_i, $i < b(k)$, and more than k structures taken from the set $\{A_{b(k)}, A_{b(k)+1}, \ldots\}$; because of this property A^* has both $L^{\omega}_{\infty,\omega}$ and MSO 0-1 laws. A^* also clearly has a recursive measure. However, it is fairly easy to see that MSO does not reduce to first-order on any subset of A^* of measure 1. To see this, let θ be a sentence of MSO with no probability on A. Let $\psi(x)$ be a formula which says "there is a connected set U with diameter 2 such that $x \in U$ and θ relativized to U is true". The truth of $\psi(x)$ depends only on the isomorphism class of the connected component that x is in. Suppose that $\psi(x)$ is equivalent to some first-order formula $\sigma(x)$ on some subset S of A^* of measure 1. Then the sentence $\exists x \sigma(x)$ (perhaps slightly modified to avoid vacuous quantification) will be true on a structure A_i in A, $i \geq b(3)$, if and only if θ is true in that structure, and thus it will not have a probability. But A has an $L^{\omega}_{\infty,\omega}$ 0-1 law, and so $\exists x \sigma(x)$ must have probability either 0 or 1, a contradiction.

Note that the first-order random theory of A^*, like that for equivalence classes, has finitely many L^k types for each k but infinitely many 1-types, and thus is not $\aleph_0$-categorical. Thus the following modification of the original conjecture may still be true.

5.15 0-1 laws and collaps to FO: II — *Monica McArthur*

Let C be a class of finite models whose first-order random theory is $\aleph_0$-categorical. Then C has an MSO 0-1 law if and only if MSO reduces to FO on C with probability 1.

5.16 0-1 laws and collaps to FO: III — *Jurek Tyszkiewicz*

Concerning previous 2 problems, it is still the case, that there are many examples, when the first-oder random theory is not $\aleph_0$-categorical, and yet the hypothesis of both conjectures is true. So the noncategorical case is also likely to have a true modification. The question is: how to modify it?