Infinite Two-Player Games with Partial Information: Logic and Algorithms

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Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine weiteren als die angegebenen Quellen und Hilfsmittel verwendet sowie Zitate kenntlich gemacht habe.

Aachen, den   2008

(Bernd Puchala)

Contents

Chapter 1

Introduction

Game theory is a mathematical field which has attracted many scientists since the seminal work of John von Neumann and Oskar Morgenstern, [vNM44]. It is concerned with the interaction of several participants, for example customers in a bargain, the opponents of a political conflict or processes in a computing system. Game theory aims on modeling situations where the behaviour of the participants is hard to predict in order to contribute to analyzing and understanding the situation by means of general results. The theory is well developed with strong results in many directions and has various applications. Probably the most well-known example of a deep result from game theory is Nash's Theorem which says that each strategic game (with finitely many actions for each player and von Neumann preferences over mixed strategy profiles) has a mixed strategy Nash-equilibrium. Another example is Martin's Theorem which says that each Borel-game is determined.

In the so called classical game theory with applications in economy, biology and social behaviour, partial information plays an important role. In particular, each strategic game is a game with partial information because all players choose their action simultaneously, and so none of the players knows which actions the other players choose when he chooses his own action. This leads to the notion of a Nash-equilibrium, where each player is assumed to behave rational and is assumed to believe that any other player behaves rational and so on. Such a game can be seen as a game with partial information where the players choose their actions sequentially, but no player is informed about the actions chosen by the other players. Such a lack of information which concerns only the past moves of the game is called imperfect information. Usually, imperfect information is considered in extensive games where the game is given by a game tree. Imperfect information is then defined via an equivalence relation on the nodes of the tree for each of the players. We will return to this concept in Chapter 2. Special solution concepts for such games have been developed, like for example, sequential equilibria where the beliefs of the players about the other player's actions are part of the equilibrium.

On the other hand, in games with incomplete information, the players are not only uncertain about the past moves of the game but also about the characteristics of the game, like the payoff that they receive if a certain strategy profile is established. In classical game theory, this is modeled by so-called Bayes-games. In such games, the actual game is unknown, and each player has a certain belief about the probabilities with which each of the possible games is the actual game. When a game is chosen, each player receives a signal which provides him a subset of all the possible games, so that he knows that each game which is not contained in the subset is not the actual game. This yields a new probability distribution for each player which is computed according to Bayes law. For an overview over classical game theory and partial information in games, see for example [Osb03] or [OR94].

Classical game theory is mostly concerned with games which end after a finite number of rounds (often even after a single round) or with situations where such a finite game is repeated infinitely often. Given its applications, this approach is very natural. However, in applications from mathematics and computer science, infinite games arise. Such games are played on directed graphs where a token is moved along the edges of the graph. Usually, finite or finitely represented (for example generated by a push-down system) graphs are considered. While model checking games for first-order logic and modal logic as well as Ehrenfeucht-Fraïssé-games are still finite games, model checking games for fixed point logics and games modeling nonterminating reactive systems are infinite, that means, an infinite number of rounds is played. In each round, an action is chosen by one of the players. This action determines the next position of the game, and an infinite sequence of positions and actions is called a play of the game and yields a payoff for each player of the game. In applications in computer science, the payoff of a play is usually -1 or 1, so that a player either loses the play or wins the play. The area of two-player win-loss games with full information is quite well developed. Deep results and good algorithms have been established for these games, especially concerning the existence and the problem of finding winning strategies for one of the players.

However, modern applications in computer science and mathematics often involve extended models, like multiplayer games and games with partial information. For example, considering a distributed system, we are interested in the question whether the multiple components of the system can cooperate such that a common goal is achieved. For a survey on recent research about infinite multiplayer games see for example the Diploma thesis on which [Umm06] is based. Furthermore, it is not always realistic to assume that the players in a game have full information. For example, in a distributed system, the communication between the components is usually bounded. Therefore, some components may not know everything about the other components. In particular, if the communication is bounded, common knowledge cannot be achieved in general. Of course we have to define an appropriate model of *bounded communication of processes* and of *knowledge* in such a system. A model for systems with multiple processes which can communicate in a bounded fashion is for example given by asynchronous message passing systems, see the book [FHMV03]. There it is shown that common knowledge cannot be gained in such systems (and that it can neither be lost). An excellent overview over knowledge in computer science systems can also be found in this book. We will return to this issue in Chapter 2.

Other examples of applications of games with partial information in computer science and mathematics are games which model the interaction between a controller and a system, where the controller does not exactly know the recent state of the system, and model checking games for independence logics. Independence logics are logics where quantification can be made independent of previous choices. Independence friendly logic has been introduced by Hintikka and Sandu in [HS96]. The model checking games for these logics are games with partial information. In [Sev06], independence logics and certain connections to games with imperfect information are studied. In [Bra00], Bradfield considers independence friendly logic in a modal logic setting. In [Bra03], he uses parity games with imperfect information as model checking games for a fixed point extension of an independent modal logic.

A *controller with partial information* should be able to force the system into a legal behaviour, but the system delivers unprecise information about its state to the controller or it has certain private states which the controller cannot see. The games which model this problem are two-player games with partial information, and we are interested in finding a winning strategy for one of the players. If we make some minor assumptions about the game, then the existence of a winning strategy for player $i \in \{0, 1\}$ is equivalent to the existence of a strategy for player i which is winning against all strategies of player $1 - i$. (Notice that we are talking about *partial information strategies.*) Furthermore, incomplete information and imperfect information coincide in this context. We will return to this point in Chapter 2. Because of this, we shall simply speak of *two-player games with partial information.*

Unless explicitly mentioned otherwise, throughout this thesis we consider two-player games with partial information.

These games and the problem of finding a winning strategy for one of the players have been considered in [Rei79] and [Rei84] in the context of computing systems and computational complexity. More recent research on this topic (and some related topics) can be found for example in [CDHR06] (and some related papers like [dWDHR06]) and in several papers of Kupferman and Vardi like [KV99], [KV97] and [vdMV98]. Furthermore, in [AHK02], alternating time temporal logic is used to express certain properties of games with partial information. The work in [AVW03] is also still related to the strategy problem for two-player games with partial information. In particular, the notion of *unobservable events* is related to the notion of *hidden private moves* that we consider in this thesis.

There has also been some work on games with both multiple players and partial information for example in [PRA01], [PRA02] and in [AHK02]. In [PRA01] it is shown that the following decision problem is undecidable. Given a finite three-player game with partial information, an initial position v_0 and a set R of positions, are there strategies f_0 for player 0 and f_1 for player 1 such that any play from v_0 that is compatible with f_0 and f_1 finally reaches R? A varied model which is suited for model checking a certain logical language is used in [Kai06]. There, the set of players is partitioned into two coalitions, each of which consists of k players. A player with number i (in either of the coalitions) can see the moves of all players with numbers $j \leq i$ in both coalitions but he cannot see the moves of players with numbers $j > i$. Strategies are chosen "alternatingly" so that committing on strategies is compatible with the knowledge of the players. It is shown that in general these games are not determined and that it is not possible to decide whether coalition I has a winning strategy. But if one assumes that the players move strictly alternating, then the games become determined and deciding the winner is possible.

Finally, the work of van Benthem and other researchers associated with him, has a great influence on current research on knowledge in multiagent systems, particularly games. The focus of their research is on dynamic epistemic logics, that means, logics in which one can describe the effect of actions on the knowledge of the agents in any kind of multiagent system. Interesting connections to games can be found for example in [vB01], [vBL04] and in [vD00].

In particular, [vD00] provides a good introduction into dynamic epistemic logic in the context of games. Knowledge games have been designed to model the game "Cluedo", a card game where cards are not interchanged, but only knowledge evolves through questions that the players ask each other. There are multiple players in these games and determining the deal of cards is a kind of probability move. But if we look at the two-player case and let one of the players choose the deal of cards, then these games can be represented by the model that we use here. Of course it has to be mentioned that the structures which are used to model knowledge games are dynamic, that means, the structures are affected by the actions which the players choose. On the contrary, the structures that we use in this thesis are static, and the dynamic aspect of a game is covered by considering a certain play of the game. In [vB01], van Benthem says: "Game trees are static pictures of all that can happen in a game. But in any specific run, there is an actual course of events, taking players to successive game states, each with different knowledge and ignorance."

Outline

In Chapter 2 we introduce and discuss a general model for games with partial information. We derive some simple results and we show that the models from [Rei79] and [CDHR06] are special cases of this model.

Chapter 3 deals with the strategy problem for games with partial information. We consider two notions of strategies which have been used by Reif in his papers [Rei79] and [Rei84]. We call them *strategies* in the standard case and *strategies if private moves are hidden* in the extended case. In Section 3.1 we adapt Reif's powerset construction from [Rei84] to our model in order to solve the strategy problem for a certain class of games with partial information in exponential time and we generalize this construction in two directions. First, the game graphs are not assumed to be finitely branching, and second, we do not make any assumptions about the winning conditions. Hereafter, we describe the winning condition of the resulting game more concrete for certain special cases like observation based winning conditions and omega-regular winning conditions on finite game graphs. In particular, we show that omega-regular winning conditions are transformed into omega-regular winning conditions by this construction.

In Section 3.5 we present a detailed version of Reif's proof for the EXPTIME-hardness of the strategy problem for games with partial information where private moves are hidden, see [Rei84]. In Section 3.4 we modify the powerset construction in order to solve the strategy problem for omega-regular games with partial information for the case where private moves are hidden. We introduce strongly observation based winning conditions and we consider information compatible Muller-games as a special case. We present a polynomial time reduction of the strategy problem for information compatible Muller-games if private moves are hidden to the (usual) strategy problem for this class of games. This shows that both problems are EXPTIME-complete. In Section 3.7, we adapt one of the main results from [CDHR06] to our model.

Furthermore, we make some contributions to aspects of games with partial information that have not been considered very much so far, like finite memory strategies and the intimate connection between games with partial information and universal tree automata. We prove upper and lower bounds for the amount of memory which is needed to win in certain classes of games with partial information, and we show how nonemptiness of alternating tree automata can be checked using games with partial information.

In Chapter 4 we consider the definability of the two equivalence relations on finite play prefixes that we use in this thesis in certain logical systems. We show that the equivalence relations can be defined in LFP and GSO, while they can neither be defined in the (bidirectional) two-dimensional μ-calculus, nor in MSO. This analysis aims on understanding the objects that we have defined and on preparing generalizations of this setting, in which for example games are considered where the equivalence relation on finite play prefixes is given by a formula from an appropriate logical system.

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Chapter 2

The Model

2.1 Games

Definition 2.1. A (*deterministic*, *turn based*) *game* is given by a tuple $G = (N, V, (f_a)_{a \in A}, (V_i)_{i \in N}, \lambda)$ with the following components.

(1) $N \neq \emptyset$ is the finite set of players.
(2) $V \neq \emptyset$ is the set of positions.
(3) $A \neq \emptyset$ is the set of actions and for $a \in A$, $f_a : \mathrm{dom}(f_a) \subseteq V \to V$ is a function.
(4) $(V_i)_{i \in N}$ is a partition of the set V. For $i \in N$, V_i is the set of positions of player i.
(5) $\lambda : S(V_{in}) := \bigcup\{v(AV)^\omega \mid v \in V_{in}\} \to \mathbb{R}^N$ for some $V_{in} \subseteq V$ is the payoff function.

For each $v \in V$ the set $\mathrm{act}(v) = \{a \in A \mid v \in \mathrm{dom}(f_a)\}$ of actions that are available at position v has to be nonempty. V_{in} is called the set of possible initial positions. For $i \in N$ we define $A_i := \bigcup\{\mathrm{act}(v) \mid v \in V_i\}$. Notice that the sets A_i for $i \in N$ are not necessarily disjoint.

For $v \in V_{in}$, $P(v)$ is the set of plays in G from initial position v where a *play* in G from initial position v is an infinite sequence $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in v(AV)^\omega$ such that $v_i \in \mathrm{dom}(f_{a_i})$ and $f_{a_i}(v_i) = v_{i+1}$ for each $i < \omega$. For $i < \omega$ we define $\pi(i) := a_i v_{i+1}$ and we define $\pi(\leq i) := \mathrm{first}(\pi)\pi(0) \ldots \pi(i)$.

The directed graph (V, E) with $E = \bigcup\{E_a \mid a \in A\}$ and $E_a = \{(u, v) \in V \times V \mid u \in \mathrm{dom}(f_a) \text{ and } f_a(u) = v\ \}$ for each $a \in A$ is called the underlying *game graph* of G. A game is called an extensive form game, if the underlying game graph is a (directed, rooted) tree and $V_{in} = \{v_0\}$, where v_0 is the root of the tree.

Remark. Usually we define the payoff function λ by prescribing a value for each *play* of the game. In many interesting cases the function is implicitly defined on sequences from $S(V_{in})$ which are not plays. For example if we price the positions of the game with nonnegative real values and let the value of λ on a play be the infimum over all occurring numbers. In all the other cases we usually ignore the value of λ on sequences which are not plays.

Definition 2.2. Let $G = (N, V, (f_a)_{a \in A}, (V_i)_{i \in N}, \lambda)$ be a game and for a position $v \in V_{in}$ let $P_{\text{fin}}(v)$ be the set of all nonempty finite prefixes $\pi \in v(AV)^*$ of plays in G from v. Furthermore for $V' \subseteq V_{in}$ let $P_{\text{fin}}(V') = \bigcup\{P_{\text{fin}}(v) \mid v \in V'\}$ be the set of all nonempty finite prefixes of plays in G from some initial position $v \in V'$. For $\pi \in V(AV)^* \cup V(AV)^\omega$, by $l(\pi) \leq \omega$ we denote the number of positions in π.

A *strategy* for player $i \in N$ for G from initial positions in V' is a function $g : \{\pi \in P_{\text{fin}}(V') \mid \text{last}(\pi) \in V_i\} \to A$ such that $g(\pi) \in \text{act}(\text{last}(\pi))$ for all $\pi \in \text{dom}(g)$. A prefix $\pi = v_0 a_0 v_1 \dots$ of a play in G from some initial position $v_0 \in V'$ is called *compatible* with g if for all $j < l(\pi)$ such that $v_j \in V_i$ we have $a_j = g(v_0 a_0 \dots a_{j-1} v_j)$. A strategy g for player i is called *positional* if $g(v_0 a_0 \dots a_j v_{j+1}) = g(v'_0 a'_0 \dots a'_k v'_{k+1})$ for all $v_0 a_0 \dots a_j v_{j+1}, v'_0 a'_0 \dots a'_k v'_{k+1} \in \text{dom}(g)$ with $v_{j+1} = v'_{k+1}$. A positional strategy g can be written as a function $g : V_i \to A$.

Definition 2.3. Let $G = (N, V, (f_a)_{a \in A}, (V_i)_{i \in N}, \lambda)$ be a game. A *memory structure* for G is given by a triple $M = (S, \delta_0, \delta)$ where S is a set of states, $\delta_0 : V' \to S$ for some $V' \subseteq V_{in}$ is the initializing function and $\delta : S \times (A \times V) \to S$ is the update function.

A *memory strategy* for player $i \in N$ for G with respect to M is a function $g : S \times V_i \to A$ such that $g(s, v) \in \text{act}(v)$ for all $(s, v) \in S \times V_i$. A prefix $\pi = v_0 a_0 v_1 \dots$ of a play in G from some initial position $v_0 \in V'$ is called *compatible* with g if for all $j < l(\pi)$ such that $v_j \in V_i$ we have $a_j = g(\delta^*(v_0 a_0 v_1 \dots a_{j-1} v_j), v_j)$. Where $\delta^* : P_{\text{fin}}(V') \to S$ is inductively defined by $\delta^*(v) = \delta_0(v)$ for $v \in V'$ and $\delta^*(v_0 a_0 \dots a_j v_{j+1}) = \delta(\delta^*(v_0 a_0 \dots a_{j-1} v_j), (a_j, v_{j+1}))$ for $v_0 a_0 \dots a_j v_{j+1} \in P_{\text{fin}}(V')$.

Notice that positional strategies are exactly those strategies that can be implemented by a memory structure with a single state. Therefore they are often called memoryless strategies. Furthermore notice that we allow arbitrary sets of states for a memory structure and so in particular we can use $P_{\text{fin}}(V_{in})$ as set of states. Thus, memory strategies are a generalization of strategies. But of course we only talk about memory strategies, if we have a simpler memory than $P_{\text{fin}}(V_{in})$. Particularly, *finite* memory strategies are of interest.

2.2 Knowledge in Multiagent Systems

Jaakko Hintikka's seminal work [Hin62] is the major starting point for the logical analysis of knowledge and belief. There, Hintikka uses modal logic to formulate and reason about certain rules for the notions of knowledge and belief. This modal logic provides modal operators K_a where a refers to a person and $K_a p$ for a certain sentence p means "a knows that p". To reason formally about knowledge and belief, Hintikka uses *model sets* which are sets of modal sentences that satisfy certain rules of consistency (Hintikka calls them rules of *defensibility*) and *model systems* which are sets of model sets. On a model system, for a person referred to by a, an *alternative relation* (more precisely an *epistemic alternative relation*, distinguishing it from the *doxastic alternative relation*) is defined, which interacts with the modal operator K_a in such a way that the sentence $K_a p$ is in a model set μ if and only if p belongs to any alternative μ^* to μ with respect to a. This is the precise sense of "a

knows that p". Any model set in a model system represents a (partial) description of a possible state of affairs. Now in the "world" μ, the person referred to by a knows the fact p, if p holds in any world, that p considers possible, that means, in all the alternatives to μ with respect to a. This is called the possible worlds model. The alternative relation is reflexive and transitive, but it is *not symmetric*. This is the main difference of Hintikka's approach compared to modern models of knowledge in computer science. Without symmetry, the negative introspection axiom does not hold in general, that means, there are sentences p, such that $\neg K_a p \to K_a \neg K_a p$ is not a valid sentence. Hintikka explains this as follows."Now it is obviously not excluded by what I now know that I should know more than I now do. But such additional knowledge may very well be incompatible with what now is still possible, as far as I know." This is a very evident argumentation and the reason that in modern applications in computer science, symmetry is assumed is not that this argument has been refuted. It is merely due to the fact that Hintikka reasons about human knowledge in its full generality while in many concrete applications, the knowledge of the agents in a system is defined via a fixed amount of information that any agent has. This information may change over time, but at any point in time, it completely determines the knowledge of an agent. The "alternative relations" that arise in such models are equivalence relations intrinsically. To make this more precise, at this point we shall give a formal definition of knowledge in multiagent systems. We closely follow [FHMV03] and we shall be as brief as possible. For an in depth introduction and an excellent overview over certain aspects of this interesting field we refer to [FHMV03].

We capture Hintikka's idea of "possible worlds" by Kripke-structures. A Kripke-structure is simply a directed graph with labelled edges and a set $\Phi_v \subseteq \Phi$ of atomic propositions assigned to any vertex v, where Φ is an arbitrary set of atomic propositions, which contains all the relevant *basic* facts about the structure. In a game for example we should have propositions like "it is player i's turn" and "action a is available to player i" and so on. The labels on the edges are taken from the set $\{1, \dots, n\}$ for some $n < \omega$ where each $i \in \{1, \dots, n\}$ represents an agent in the system. For $i \in \{1, \dots, n\}$, the edge relation E_i defines the knowledge of agent i, that means, $K_i\varphi$ for some formula φ of modal logic holds at a vertex v, if and only if φ holds at any vertex w with $(v, w) \in E_i$. (We assume that the reader is familiar with syntax and semantics of basic modal logic. Notice that the operator K_i is usually written as $[i]$.) Now let $\mathcal{M}_n$ be the class of all Kripke-structures with n agents, let $\mathcal{M}_n^\zeta$ for $\zeta \in \{s, r, t\}$ be the class of all such structures where all relations E_i are symmetric, reflexive and transitive respectively. We say that a formula φ is valid in a structure $\mathcal{K} \in \mathcal{M}_n$, written $\mathcal{K} \models \varphi$, if φ holds at each vertex of $\mathcal{K}$. The formula is valid in a class $\mathcal{M} \subseteq \mathcal{M}_n$, written $\mathcal{M} \models \varphi$, if it is valid in each structure from $\mathcal{M}$. For any $n < \omega$, any $i \in \{1, \dots, n\}$ and all formulas φ, ψ of modal logic, the following propositions hold.

(A1) $\mathcal{M}_n \models (K_i\varphi \land K_i(\varphi \to \psi)) \to K_i\psi$
(Distribution Axiom)

(A2) For all $\mathcal{K} \in \mathcal{M}_n$, if $\mathcal{K} \models \varphi$ then $\mathcal{K} \models K_i\varphi$
(Knowledge Generalization Rule)

(A3) $\mathcal{M}_n^r \models K_i\varphi \to \varphi$
(Knowledge Axiom)

(A4) $\mathcal{M}_n^t \models K_i\varphi \to K_iK_i\varphi$
(Positive Introspection Axiom)

(A5) $\mathcal{M}_n^s \cap \mathcal{M}_n^t \models \neg K_i\varphi \to K_i\neg K_i\varphi$
(Negative Introspection Axiom)

These five propositions are called the S5-axioms. (Notice that one can easily check that the above classes are not the largest classes in which the respective axioms are valid.) These axioms together with all tautologies of propositional calculus and modus ponens as inference rule, form a sound and complete axiom system for modal logic with respect to $\mathcal{M}_n^s \cap \mathcal{M}_n^t \cap \mathcal{M}_n^r$, cf. [FHMV03]. Furthermore, one should notice that this formalization of knowledge by means of Kripke-structures entails that the agents are *logically omniscient*, that means, they know all tautologies and they know all logical consequences of their knowledge. For a more detailled discussion of this issue, see [FHMV03].

Now we make precise what we mean by defining knowledge by means of a certain amount of information that any agent has. So consider an n-agent system $\mathcal{K}$ with atomic propositions from Φ. For any $i \in \{1, \ldots, n\}$ we fix a set $\Phi_i \subseteq \Phi$ and we define $(v, w) \in E_i$ if for all $p \in \Phi_i$ we have $p \in \Phi_v$ if and only if $p \in \Phi_w$. The set Φ_i is said to be the information of player i in the system $\mathcal{K}$. Clearly any relation E_i is an equivalence relation. So in the world v, agent i considers the world w possible, if agent i's information is insufficient to enable him to distinguish whether the actual world is v or w. In a somewhat less precise sense we usually say that the worlds v and w are indistinguishable for agent i.

Of course, this is only a snapshot of the system. An agent may gather information as time goes by and if he has only a bounded amount of memory available, then he might also lose some information. Furthermore, the actual world might change over time, which is obviously the case in almost any computing system, where the "actual world" is the recent state of the whole system, which itself may consist of many components, each of which is represented by an agent in the system. To abstract from the information sets Φ_i, we simply define a function $\mathrm{vis}_i : V \to \mathrm{VIS}_i$ for some set VIS_i, which extracts from a world $v \in V$ the information $\mathrm{vis}_i(v)$ that is visible for agent i. In the case of information sets we have $\mathrm{vis}_i(v) = \Phi_v \cap \Phi_i$.

In [FHMV03], a system is a set of runs. That means, the system is given by its possible behaviours. We assume that time is discrete and infinite, that means, it ranges over the natural numbers. This is particularly appropriate since we consider infinite games which are played in distinct rounds. So a system is a set $\mathcal{S}$ of functions $r : \omega \to V$ mapping each point $k < \omega$ of time to the state $r(k) \in V$ of the system in the run r after k time steps. For any $v \in V$ we call $\mathrm{vis}_i(v)$ the local state of agent i in $\mathcal{S}$. So $\mathrm{vis}_i(r(k))$ is the local state of agent i in the run r after k time steps. We call a pair (r, m) with $r \in \mathcal{S}$ and $m < \omega$ a point in the system. Now agent i cannot distinguish two points (r, m) and (r', m') in the system, if $\mathrm{vis}_i(r(m)) = \mathrm{vis}_i(r'(m'))$. If for all $r, r' \in \mathcal{S}$ and all $m, m' < \omega$, $\mathrm{vis}_i(r(m)) = \mathrm{vis}_i(r'(m'))$ implies $m = m'$, then the system is called synchronous. Roughly speaking the agents in a synchronous system have access to a shared clock. Now let agent i's local state sequence at the point (r, m) in the system be the sequence of local states that he has gone through in run r up to time m, without consecutive repetitions. Agent i in a system is said to have perfect recall, if for all $r, r' \in \mathcal{S}$ and all $m, m' < \omega$, $\mathrm{vis}_i(r(m)) = \mathrm{vis}_i(r'(m'))$ implies that agent i's local state sequence in (r, m) and in (r', m') coincide.

Now if a system shall be the input of an algorithm, then of course such an "extensive representation" is not appropriate, since it is infinite. So we have to describe a system in a more compact way. Rather than taking all the possible runs (behaviours) of the system as a description, we use Kripke-structures as "physical models" of systems. Of course, these Kripke-structures can also be infinite in general, for example if we consider a register-machine, where the variables of the machine are unbounded. But in many interesting cases, the structures will be finite. For example, finite automata, push-down systems and recursive automata and, most important in our case, games played on finite graphs can be represented by finite structures. (The models of push-down systems and recursive automata are not actually Kripke-structures, but they are very interesting examples of system with "infinite behaviour" which have a finite representation.) So consider any given Kripke-structure $\mathcal{K} = (V, (E_a)_{a \in A}, (P_i)_{i \in I})$ over atomic propositions from $\Phi = \{P_i \mid i \in I\}$, where now the edge relations E_a for $a \in A$ do not define knowledge of agents, but they define transitions of the system from one state to another. We partition $V = V_1 \cup \ldots \cup V_n$, where $v \in V_i$ means that at the position v, agent i chooses an action which is executed.

Now a system in the sense of [FHMV03] is given via $\mathcal{K}$ by the set $\mathcal{R}(\mathcal{K}, V_{in})$ of all possible runs of $\mathcal{K}$ from positions in a set $V_{in} \subseteq V$ of initial states. Where a run of $\mathcal{K}$ from a position $v \in V_{in}$ is an infinite sequence $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in v(AV)^\omega$ such that $(v_i, v_{i+1}) \in E_{a_i}$ for each $i < \omega$. For $i < \omega$ we define $\pi(i) := a_i v_{i+1}$ and we define $\pi(\leq i) := \mathrm{first}(\pi)\pi(0)\ldots\pi(i)$. Furthermore we define $\mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V_{in}) := \{\pi(\leq i) \mid \pi \in \mathcal{R}(\mathcal{K}, V_{in}), i < \omega\}$. The set $\mathcal{R}_{\mathrm{fin}}(\mathcal{K}, v)$ for $v \in V$ is the set of all nodes in the unravelling of $\mathcal{K}$ from v and $\mathcal{R}(\mathcal{K}, v)$ is the set of all infinite paths through this unravelling.

Of course, we do not only want a finite representation of the physical model of the system, but also a finite representation of the knowledge of the agents in the system, that means, in the set of runs of the physical model. One possibility would be to consider for each agent i a finite automaton $\mathcal{A}_i$ with output which reads a finite prefix $\pi \in \mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V)$ of a run of the system and outputs the local state of agent i in state π. Another possibility would be to define equivalence relations directly, for example by a logical formula $\varphi(x, y)$ with two free element variables, where agent i cannot distinguish two states $\pi, \pi' \in \mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V)$ of the system, if and only if $\varphi(\pi, \pi')$ holds in the structure $\mathfrak{A} = (\mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V), \tau^{\mathfrak{A}})$ for an appropriate signature τ. We could also use finite automata over relations. There are three basic ways how an automaton can read pairs of words. The first possibility is to parse a pair $(u, v) \in R$ by reading the word $u\#v$ for some special symbol $\#$. If a relation can be recognized in that way, it is called componentwise recognizable. The second possibility is to parse (u, v) by reading $(u_1, v_1)\ldots(u_n, v_n)(x_{n+1}, y_{n+1})\ldots(x_m, y_m)$ where $x_i = \#$ or $y_i = \#$ dependent on the length of u and v respectively. If a relation can be recognized in such a fashion it is called automatic. The third possibility is to allow the automaton to read in one component while not reading in the other component. If a relation can be recognized by such an asynchronous automaton it is called rational.

In this thesis, we consider two possibilities to define knowledge in a system. First we define the information that an agent is given in the physical part of the model and we keep this information fixed. So for $i \in \{1, \ldots, n\}$ we define a function $\mathrm{vis}_i^V : V \to \mathrm{VIS}_i^V$ into a set VIS_i^V and for any $v \in V$, $\mathrm{vis}_i^V(v)$ is the information that agent i has about the state v in the system $\mathcal{K}$. Furthermore, we also define the information that an agent has about an action $a \in A$ in the system $\mathcal{K}$, so for $i \in \{1, \ldots, n\}$, we define a function $\mathrm{vis}_i^A : A \to \mathrm{VIS}_i^A$ into a set VIS_i^A. For $i \in \{1, \ldots, n\}$ the functions vis_i^V and vis_i^A yield the usual equivalence relations $\sim_i^V \subseteq V \times V$ and $\sim_i^A \subseteq A \times A$ where $x \sim_i^\# y$ if and only if $\mathrm{vis}_i^\#(x) = \mathrm{vis}_i^\#(y)$. Furthermore, for $i \in \{1, \ldots, n\}$ we define $\mathrm{vis}_i : AV \to \mathrm{VIS}_i^A \, \mathrm{VIS}_i^V$ via $\mathrm{vis}_i(Av) := \mathrm{vis}_i^A(a) \, \mathrm{vis}_i^V(v)$ and we denote the corresponding equivalence relation by $\sim_i$.

Now we consider two possibilities to extend the function vis_i for $i \in \{1, \ldots, n\}$ to the set $\mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V)$. We denote the extensions by vis_i^* and vis_i^+ respectively. We call a transition (u, v) of the system insignificant for agent i, if $u \notin V_i$ and $u \sim_i^V v$. We call the transition significant, if it is not insignificant.

(∗) $\mathrm{vis}_i^*(\pi) = \mathrm{vis}_i^V(\pi)$ if $\pi \in V$

(∗) $\mathrm{vis}_i^*(\pi a v) = \mathrm{vis}_i^*(\pi) \, \mathrm{vis}_i(Av)$.

(+) $\mathrm{vis}_i^+(\pi) = \mathrm{vis}_i^V(\pi)$ if $\pi \in V$.

(+) $\mathrm{vis}_i^+(\pi a v) = \mathrm{vis}_i^+(\pi)$, if $(\mathrm{last}(\pi), v)$ is insignificant for agent i.

(+) $\mathrm{vis}_i^+(\pi a v) = \mathrm{vis}_i^+(\pi) \, \mathrm{vis}_i(Av)$ if $(\mathrm{last}(\pi), v)$ is significant for agent i.

We call $\mathrm{vis}_i^*(\pi)$ the local state of agent i in state π and we call $\mathrm{vis}_i^+(\pi)$ the local state of agent i in state π if insignificant transitions are hidden. We denote the corresponding equivalence relations by $\sim_i^*$ and $\sim_i^+$. Now let $\pi = v_0 a_0 v_1 \ldots a_{n-1} v_n \in \mathcal{R}_{\mathrm{fin}}(\mathcal{K}, V)$. It is easy to see that

(∗) $\mathrm{vis}_i^*(\pi) = \mathrm{vis}_i^V(v_0) \, \mathrm{vis}_i(a_0 v_1) \ldots \mathrm{vis}_i(a_{n-1} v_n)$ and

(+) $\mathrm{vis}_i^+(\pi) = \mathrm{vis}_i^V(v_0) \, \mathrm{vis}_i(a_{i_1} v_{i_1+1}) \ldots \mathrm{vis}_i(a_{i_k} v_{i_k+1})$,

where $v_0 a_{i_1} v_{i_1+1} \ldots a_{i_k} v_{i_k+1}$ is the sequence from $V(AV)^*$ which results from π by contracting each maximal sequence $v_r a_r v_{r+1} a_{r+1} \ldots a_s v_{s+1}$ in π such that (v_j, v_{j+1}) is insignificant for agent i for $j = r, \ldots, s$ to v_r.

In Chapter 4 we discuss in detail, for which logical systems, the (global) relations $\sim_i^*$ and $\sim_i^+$ can be defined by a logical formula $\varphi(x, y)$ (over appropriate signatures). Furthermore it is easy to see that (in the finite case) the relation $\sim_i^*$ is always automatic but not always componentwise recognizable, while $\sim_1^+$ is always rational but not always automatic.

So far, the information of an agent i about the states and the actions of the physical model have been assumed to be static. Now of course we could assume that they change while time passes by. Then we have to use dynamic visibility functions $\mathrm{visD}_i^V : (VA)^* \to (\mathrm{VIS}_i^V)^V$ and $\mathrm{visD}_i^A : (VA)^* \to (\mathrm{VIS}_i^A)^A$, that means, after each finite prefix $\pi \in \mathcal{R}_{\text{fin}}(\mathcal{K}, V)$ the information of agent i about the states and actions of the physical model are given by the functions $\text{visD}_i^V(\zeta(\pi)) : V \to \text{VIS}_i^V$ and $\text{visD}_i^A(\zeta(\pi)) : A \to \text{VIS}_i^A$ where $\zeta(\pi)$ is obtained from π by deleting the last state. Using these dynamic information we can define the local states of the agents in states $\pi \in \mathcal{R}_{\text{fin}}(\mathcal{K}, V)$ for example in the following way.

- $(*)$ $\text{vis}_i^*(\pi) = \text{visD}_i^V(\varepsilon)(\pi)$ if $\pi \in V$
- $(*)$ $\text{vis}_i^*(\pi a v) = \text{vis}_i^*(\pi)\,\text{visD}_i^A(\zeta(\pi))(a)\,\text{visD}_i^V(\pi a)(v)$

Now again we want to have a finite representation of the functions visD_i^V and visD_i^A. One possibility would be to use finite automata $\mathcal{A}_i^V = (Q^V, q_0^V, \delta^V)$ and $\mathcal{A}_i^A = (Q^A, q_0^A, \delta^A)$ and functions $\zeta^V : Q^V \times V \to \text{VIS}_i^V$ and $\zeta^A : Q^A \times A \to \text{VIS}_i^A$. Then for $\pi \in \mathcal{R}_{\text{fin}}(\mathcal{K}, V)$ the function $\text{visD}_i^V(\pi)$ coincides with the function $\zeta^V((\delta^V)^*(\pi), \cdot)$ and analog for visD_i^A. Now if we use this possibility, then we can reduce the "dynamic case" to the "static case", by taking the product of $\mathcal{K}$ with all the automata $\mathcal{A}_i^V$ and $\mathcal{A}_i^A$ for $i \in \{1, \dots, n\}$ and defining the information of an agent i about a state $\overline{v} = (v, q_1^V, \dots, q_n^V, q_1^A, \dots, q_n^A)$ by $\text{vis}_i^V(\overline{v}) = \zeta^V(q_i^V, v)$. Furthermore we define $\text{act}(\overline{v}) = \{\overline{a} = (a, q_1^A, \dots, q_n^A) \mid a \in \text{act}(v)\}$ and $\text{vis}_i^A(\overline{a}) = \zeta^A(q_i^A, a)$. Since we do not consider the dynamic case any further in this thesis we do not provide a complete description of this construction. But this already shows that the model which we use is quite general. More evidence concerning this point can be found in the Sections 2.5 and 2.8.

2.3 Games with Partial Information

Now we consider partial information in games, so let $G = (N, V, (f_a)_{a \in A}, (V_i)_{i \in N}, \lambda)$ be a game. The knowledge of a player $i \in N$ after some prefix $\pi \in P_{\text{fin}}(V_{in})$ has been played is given by his local state $\text{vis}_i(\pi)$ which defines the equivalence relation $\sim_i \subseteq P_{\text{fin}}(V_{in}) \times P_{\text{fin}}(V_{in})$ where $\pi \sim_i \pi'$ if $\text{vis}_i(\pi) = \text{vis}_i(\pi')$. Now the point of the knowledge of a player is that he cannot make his choices in the game depending on facts that he does not know. That means, a strategy for a player has to be compatible with his knowledge. For example in a card game, player A cannot commit himself on a behaviour like "after 5 rounds, if player B holds the blue card, then I draw a card, otherwise, I do not draw a card", if after 5 rounds player A might not know whether player B holds the blue card. Precisely this means the following.

Definition 2.4. Let $G = (N, V, (f_a)_{a \in A}, (V_i)_{i \in N}, \lambda)$ be a game, let $V' \subseteq V$, $i \in N$ and let $\sim_i$ be an equivalence relation on the set $P_{\text{fin}}(V')$. A *partial information strategy* for player i for G with respect to $\sim_i$ from initial positions in V' is a strategy $g : \{\pi \in P_{\text{fin}}(V') \mid \text{last}(\pi) \in V_i\} \to A$ for player i for G from initial positions in V' such that $g(\pi) = g(\pi')$ for all $\pi, \pi' \in \text{dom}(g)$ with $\pi \sim_i \pi'$.

Now we consider the two possibilities to define the local state of the players in the game that we have presented in Section 2.2. We add the information that each player has about the positions and the actions of a game as an additional component to the model and we call this the epistemic component of the game. We consider only two-player games, that is, we assume $N = \{0, 1\}$. Since the set N is fixed we omit this component in the description of a game. Clearly we have $V_1 = V \setminus V_0$, so we omit V_1 as well. Finally, in the following we do not restrict the possible initial positions a priori, so we have $V_{in} = V$.

Definition 2.5. A *two-player game with partial information* has the form $\mathcal{G} = (G, (\mathrm{vis}_i^V)_{i=0,1}, (\mathrm{vis}_i^A)_{i=0,1})$ where $G = (V, V_0, (f_a)_{a\in A}, \lambda)$ is a two-player game and for $i = 0, 1$, $\mathrm{vis}_i^V : V \to \mathrm{VIS}_i^V$ and $\mathrm{vis}_i^A : A \to \mathrm{VIS}_i^V$ are functions into sets VIS_i^V and VIS_i^A respectively, such that the following conditions hold for $i = 0, 1$.

(C1) If $u, v \in V$ with $\mathrm{vis}_i^V(u) = \mathrm{vis}_i^V(v)$ then $u, v \in V_i$ or $u, v \notin V_i$.

(C2) If $a, b \in A_i$ with $a \neq b$ then $\mathrm{vis}_i^A(a) \neq \mathrm{vis}_i^A(b)$.

G is called the *physical component* of the game and $((\mathrm{vis}_i^V)_{i=0,1}, (\mathrm{vis}_i^A)_{i=0,1})$ is called the *epistemic component* of the game.

Condition (C1) says that a player always knows when it is his turn. Condition (C2) says that a player can distinguish all the actions that are available to him at some position of the game. Those conditions are quite reasonable. In particular, if (C1) would not hold, then it would be absolutely not clear, how a player should play the game. Notice that we do not require that a player knows which actions are available to him when it is his turn. The reason is that this might change over time. It is possible that a play visits a state for the first time and the player who's turn it is does not know whether action a is available and when the play visits the position for the second time, then he does know (or vice versa).

Now let vis_i^* and vis_i^+ be the local state functions that we have defined in Section 2.2 and let $\sim_i^*$ and $\sim_i^+$ be the corresponding equivalence relations. Furthermore let $\sim_i^V$, $\sim_i^A$ and $\sim_i$ be the equivalence relations on V, A and AV respectively, as we have defined them in Section 2.2. From now on, we call a move in the game which is insignificant for player i a private move of player $1 - i$.

Definition 2.6. Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ be a game with partial information, let $V' \subseteq V$ and let $i \in \{0, 1\}$. A *strategy* for player i for $\mathcal{G}$ from initial positions in V' is a partial information strategy for player i for G with respect to $\sim_i^*$ from initial positions in V'. A *strategy* for player i for $\mathcal{G}$ from initial positions in V' *if private moves are hidden* is a partial information strategy for player i for G with respect to $\sim_i^+$ from initial positions in V'.

Definition 2.7. Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ be a game with partial information, let $V' \subseteq V$ and let $i \in \{0, 1\}$. A *memory structure* for player i for $\mathcal{G}$ is a memory structure $M = (S, \delta_0, \delta)$ for G such that the following conditions hold.

(1) $\delta_0(v) = \delta_0(w)$ for all $v, w \in V'$ with $v \sim_i^V w$.

(2) $\delta(s, (a, v)) = \delta(s, (b, w))$ for all $s \in S$ and all $(a, v), (b, w) \in A \times V$ with $Av \sim_i bw$.

M is called a *memory structure* for player i for $\mathcal{G}$ *if private moves are hidden*, if additionally, the following conditions hold.

(3) If $\pi, \pi av \in P_{\mathrm{fin}}(\mathrm{dom}(\delta_0))$ such that $(\mathrm{last}(\pi), v)$ is a private move of player $1 - i$, then $\delta^*(\pi av) = \delta^*(\pi)$.

A *memory strategy* for player i for $\mathcal{G}$ with respect to M is a memory strategy $g : S \times V_i \to A$ for player i with respect to M such that for all $s \in S$ and all $u, v \in V$ with $u \sim_i^V v$ we have $g(s, u) = g(s, v)$.

A strategy for a game $\mathcal{G}$ with partial information is called positional, if it is memoryless, that means, if it is a memory strategy with respect to some memory structure for $\mathcal{G}$ which has only a single state. Notice that according to this definition, a positional strategy for a game with partial information is a positional strategy which is also a partial information strategy. The converse, however, is not true in general. That means, a positional strategy which is also a partial information strategy is not a positional partial information strategy in general.

2.4 Win-Loss Games and Winning Strategies

Definition 2.8. Let $G = (N, V, (f_a)_{a\in A}, (V_i)_{i\in N}, \lambda)$ be a game. G is called a *zero-sum game* if for all $\pi \in P(V_{in})$ we have $\sum_{i\in N} \lambda(\pi)(i) = 0$. G is called a *win-loss game* if $\lambda(P(V_{in})) = \{-1, 1\}^N$.

So in a win-loss game the payoff function is given by a family $(W_i)_{i\in N}$ of sets $W_i \subseteq \{v(AV)^\omega \mid v \in V_{in}\}$ where for $\pi \in P(V_{in})$ we define $\pi \in W_i$ if and only if $\lambda(\pi)(i) = 1$. If $\pi \in W_i$ we say that π is won by player i in G. We call the set W_i for $i \in N$ the winning condition of player i. Notice that a two-player zero-sum game is a win-loss game “up to scaling”. For any play π of such a game we have $\lambda(\pi)(1) = -\lambda(\pi)(0)$. Therefore, two-player zero-sum win-loss games are usually called two-player zero-sum games. If G is a two-player zero-sum game then λ is given by the winning condition $W_0 \subseteq \bigcup\{v(AV)^\omega \mid v \in V_{in}\}$ of player 0, since $W_1 = \bigcup\{v(AV)^\omega \mid v \in V_{in}\} \setminus W_0$.

Definition 2.9. Let $G = (N, V, (f_a)_{a\in A}, (V_i)_{i\in N}, (W_i)_{i\in N})$ be a win-loss game. We call a strategy $g : \{\pi \in P_{\mathrm{fin}}(V') \mid \mathrm{last}(\pi) \in V_i\} \to A$ for player $i \in N$ from initial positions in V' a *winning strategy* from initial position $v_0 \in V$ if $v_0 \in V'$ and each play $\pi \in P(v_0)$ of G from initial position v_0 that is compatible with g is won by player i. For $i \in N$, the *winning region* Win_i^G of player i in G is the set of all positions $v \in V$ such that player i has a winning strategy for G from v.

If $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a\in A}, W_0)$ is a game with partial information, then for $i \in \{0, 1\}$ the winning region $\mathrm{Win}_i^{\mathcal{G}}$ of player i in $\mathcal{G}$ is the set of all positions $v \in V$ such that player i has a winning strategy for $\mathcal{G}$ from v. The winning region $\mathrm{Win}_i^{\mathcal{G},h}$ of player i in $\mathcal{G}$ if private moves are hidden is the set of all positions $v \in V$ such that player i has a winning strategy for $\mathcal{G}$ from v if private moves are hidden.

Definition 2.10. A two-player zero-sum game $G = (V, V_0, (f_a)_{a\in A}, W_0)$ is called *determined* if $V = \mathrm{Win}_0^G \cup \mathrm{Win}_1^G$. The game is called *determined with memory* κ for some cardinal number κ, if it is determined and for each $i \in \{0, 1\}$ and any position $v \in \mathrm{Win}_i^G$, there is a memory structure $M = (S, \delta_0, \delta)$ for G with $|S| \leq \kappa$, such that player i has a memory winning strategy with respect to M for G from v. The game is called *uniformly determined with memory* κ if it is determined and for each

$i \in \{0,1\}$ there is a memory structure $M = (S, \delta_0, \delta)$ for G with $|S| \leq \kappa$ such that player i has a memory strategy with respect to M for G which is a memory winning strategy from each $v \in \mathrm{Win}_i^G$.

Determinacy (with memory, respectively) for games with partial information and determinacy (with memory, respectively) for games with partial information if private moves are hidden is defined completely analog.

Now we introduce one of the most important decision problem in the context of win-loss games, the so called strategy problem.

The Strategy Problem. Let $\mathfrak{G}$ be a class of finitely representable win-loss games. The strategy problem for $\mathfrak{G}$ is the following decision problem.

- Given a game $G \in \mathfrak{G}$, a player $i \in N$ and a position $v \in V$.
- Does player i have a winning strategy for G from initial position v?

Remark. Here, we restrict our attention to two-player zero-sum games and winning strategies for player 1. So, given a class $\mathfrak{G}$ of finitely representable two-player zero-sum games, the strategy problem for $\mathfrak{G}$ asks, given a game $G \in \mathfrak{G}$ and a position $v \in V$, whether player 1 has a winning strategy for G from v.

The strategy problem for partial information games and the strategy problem for partial information games if private moves are hidden are defined completely analog.

Proposition 2.1. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a game with partial information, let* $v_0 \in V$ *and let* f *be an arbitrary strategy for player* 1 *for* G *from* v_0*. Now assume that for all* $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ *with* $\pi \sim_0^* \pi'$ *we have* $\mathrm{act}(\mathrm{last}(\pi)) = \mathrm{act}(\mathrm{last}(\pi'))$*. Then the following statements are equivalent.*

(1) f *is a winning strategy for player* 1 *from* v_0*.*
(2) For any strategy g *for player* 0 *for* $\mathcal{G}$ *from* v_0*, the unique play of* G *from* v_0 *which is compatible with* f *and* g *is won by player* 1*.*

Proof. The implication from (1) to (2) is obvious, so assume conversely that (2) holds and let $\tilde{\pi} = v_0 a_0 v_1 a_1 \ldots$ be an arbitrary play of G from v_0 that is compatible with f. Furthermore let $X := \{ v_0 a_0 \ldots a_{i-1} v_i \mid i < \omega,\ v_i \in V_0 \}$. We define the strategy g for player 0 for $\mathcal{G}$ from v_0 as follows. For any equivalence class $[\pi]_{\sim_0^*} \subseteq P_{\mathrm{fin}}(v_0)$ of finite prefixes with respect to $\sim_0^*$ we define the value of g on $[\pi]_{\sim_0^*}$ as follows. If $[\pi]_{\sim_0^*} \cap X \neq \emptyset$, then clearly $[\pi]_{\sim_0^*} \cap X = \{v_0 a_0 \ldots a_{i-1} v_i\}$ for some $v_0 a_0 \ldots a_{i-1} v_i \in X$. (Of course, player 0 knows the number of moves that he has made up to a certain point in a play, so he can distinguish each two elements from X.) We define $g([\pi]_{\sim_0^*}) := \{a_i\}$, which is well defined according to the assumption about actions that we have made. If $[\pi]_{\sim_0^*} \cap X = \emptyset$, then we define $g([\pi]_{\sim_0^*}) := \{a\}$ for some action $a \in \mathrm{last}(\pi)$. Again, this is well defined according to our assumption about actions. So g is a strategy for player 0 for $\mathcal{G}$ from v_0 and $\tilde{\pi}$ is compatible with f and g. According to (2), $\tilde{\pi}$ is won by player 1 and so we have shown that f is a winning strategy for player 1 for G from v_0. □

Remark. Notice that the strategy f is assumed to be a strategy for G and not for $\mathcal{G}$, while g is assumed to be a strategy for $\mathcal{G}$. So this proposition shows that under some reasonable assumption, the existence of a winning strategy for player 1 for $\mathcal{G}$ (G respectively) from v_0 is *equivalent* to the existence of a strategy for player 1 for $\mathcal{G}$ (G respectively) from v_0 which is winning against all strategies of player 0 for $\mathcal{G}$ from v_0.
Furthermore notice that the proof works for any game G and any equivalence relation $\sim'_0$ on $P_{\text{fin}}(v_0)$ such that for all $\pi = v_0 a_0 \dots a_{n-1} v_n, \pi' = w_0 b_0 \dots b_{m-1} w_m \in P_{\text{fin}}(v)$ with $\pi \sim'_0 \pi'$, the following two conditions hold.

(1) $\text{act}(v_n) = \text{act}(w_m)$ and

(2) $|\{k \in \{0, \dots, n\} \mid v_k \in V_{1-i}\}| = |\{k \in \{0, \dots, m\} \mid w_k \in V_{1-i}\}|$.

In particular it works for the case where private moves are hidden. Of course the proof can also be extended to multiple players, where we are interested in a winning strategy for some player $i \in N$ and each equivalence relation $\sim'_j$, $j \neq i$ fulfills (1) and (2).

Now we compare the relations $\sim^*_i$ and $\sim^+_i$. It is easy to see that $\text{vis}^*_i(\pi) = \text{vis}^*_i(\pi')$ implies $\text{vis}^+_i(\pi) = \text{vis}^+_i(\pi')$ for all finite prefixes π, π' of plays in a game $\mathcal{G}$ with partial information. Therefore, a strategy if private moves are hidden is a strategy. In particular, if player i has a winning strategy for $\mathcal{G}$ from initial position $v_0 \in V$ if private moves are hidden, then he has a winning strategy for $\mathcal{G}$ from v_0. The next example shows that the converse is not true in general.

Example 2.1. *Consider the game* $\mathcal{G} = (G, (\text{vis}^V_i), (\text{vis}^A_i))$, $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *as depicted in Figure 2.1. The dotted lines define the equivalence relations* $\sim^V_1$ *and* $\sim^a_1$*, that means, the information of player* 1 *in the game. Player* 0 *wins a play of the game if one of the positions* 4 *and* 6 *is reached. Finally, circle positions belong to player* 0.

Now the function f *with* $f(0a1b2) := a$ *and* $f(0a5) := b$ *is a strategy for player* 1 *for* $\mathcal{G}$ *from* 0*, if private moves are not hidden, since* $0a1b2 \not\sim^*_1 0b5$*. And clearly* f *is a winning strategy for player* 1 *for* $\mathcal{G}$ *from* 0*. But* $0a1b2 \sim^+_1 0b5$*, since* $0 \to 1$ *is a private move of player* 0 *and* $2 \sim^V_1 5$*. In fact, player* 1 *does not have a winning strategy for* $\mathcal{G}$ *from* 0*, if private moves are hidden.*

2.5 Discussion

In this section we reflect upon the model that we have presented in the past sections. We discuss some properties and certain possible variations of the model. In particular we shall have a look at the following additional condition that we did not require in the definition of the model. In the following, by $[u]_{\sim_i}$ we denote the equivalence class of a position u with respect to $\sim^V_i$ and analog for actions.

(C3) $\text{act}(u) = \text{act}([u]_{\sim_i})$ for all $u \in V_i$.

Incomplete Information and Initial Positions. Incomplete information means that the uncertainties of the players in the game do not only concern the past moves

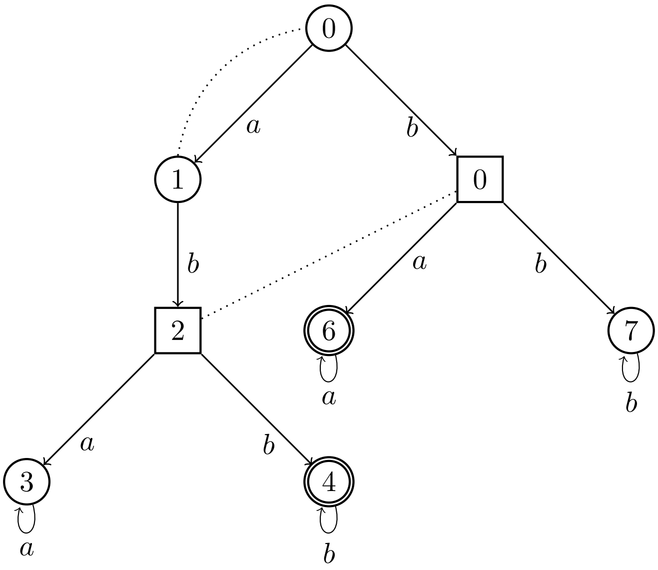

Figure 2.1: The game $\mathcal{G}$.

but also the game itself. In our case this means that the players do not necessarily know the game graph. More precisely, they do not know the connected component of the game graph in which the game is played. Of course, if player i knows the initial position v_0 of the game, then he also knows this connected component. So we have to hide v_0 from player i in order to give him incomplete information. That is, we have to say that a strategy for player i from initial position v_0 is a function $f : \{\pi \in P_{\text{fin}}(V') \mid \text{last}(\pi) \in V_i\} \to A$ for some set $V' \ni v_0$ of positions which player i should consider possible as initial positions.

Now assume that we are asking for a winning strategy of player i. We use a disjoint union of several game graphs as arena and we hide v_0 from player i. Then player i still has to take into account only play prefixes from initial position v_0, as long as he has to win only those plays which start in v_0. He can choose the value of his strategy on each play prefix which does not start in v_0 arbitrarily. Of course the strategy has to be constant over equivalence classes of play prefixes, so the action which is prescribed by player i's strategy on some prefix has to be available at the last position of each equivalent prefix as well.

So what we actually have to do in order to give player i incomplete information (if we are asking for winning strategies), is to add a new position to the game which belongs to the opponent and from which he may choose exactly the positions in V'. This is the new initial position from which we ask for a winning strategy for player i. So we can reduce the problem of finding a winning strategy in the incomplete information case to the problem of finding a winning strategy in our basic model.

Skipping Condition (C2). Consider a situation where player i can distinguish some of the actions from A_i only by their names, but he does not know their actual meaning in the game. For example if he cannot distinguish the effect of the actions $a_1, \ldots, a_n \in A_i$, then if he chooses a_i, for him it is possible that one of the actions

$a_1, \dots, a_n$ is actually executed.

Now if we ask, whether player i has a winning strategy in this setting where he cannot necessarily distinguish his own actions, then we run into a similar problem as in the previous issue. As long as player i has to win only the plays which are compatible with his strategy (that means, the plays where always the action that he chooses is actually executed), then he can define the value of his strategy on all other play prefixes arbitrarily. Of course the strategy has to be constant over equivalence classes of play prefixes, so the action which is prescribed by player i's strategy on some prefix has to be available at the last position of each equivalent prefix as well.

So in this context we actually have to say that a strategy f for player i is a winning strategy from v_0, if the following holds. Each play $\pi = v_0a_0v_1 \dots \in P(v_0)$ such that for all $j < \omega$ with $v_j \in V_i$ we have $a_j \sim_i^A f(v_0a_0 \dots a_{j-1}v_j)$ is won by player i. Now to this situation we can directly apply the powerset construction from Section 3.1. But if we assume condition (C3), then we can also transform a game where condition (C2) does not hold into a game where condition (C2) holds, such that the existence of winning strategies (in the above sense) *for player i* is preserved.

For any $u \in V_i$ we define $\mathrm{act}'(u) := \{[a]_{\sim_i} \mid a \in \mathrm{act}(u)\}$ and for any $[a]_{\sim_i} \in \mathrm{act}'(u)$ we insert a new position $(u, [a]_{\sim_i})$ of player $1 - i$ to the game. Furthermore we define $\mathrm{act}((u, [a]_{\sim_i})) := [a]_{\sim_i} \cap \mathrm{act}(u)$ and $f'_a((u, [a]_{\sim_i})) := f_a(u)$. So all the actions of the original game now belong to player $1 - i$ and the actions of player i are the sets $[a]_{\sim_i}$ for $a \in A_i$. Now we assume that player $1 - i$ has full information and the information about the new positions and actions for player i is defined via $\mathrm{vis}_i^V((u, [a]_{\sim_i})) := \mathrm{vis}_i^V(u)$ and $\mathrm{vis}_i^A([a]_{\sim_i}) := \mathrm{vis}_i^A(a)$ for some $a \in [a]_{\sim_i}$. Finally, the winning condition is defined in the obvious way. (A play in the new game is won by player i, if and only if the unique play in the old game that it 'contains' is won by player i.) It can be shown that the existence of winning strategies *for player i* is preserved by this construction.

Introducing Condition (C3). Now we discuss condition (C3) in the context of winning strategies. If the condition does not hold in some game, then for $i = 0, 1$ we do the following construction. We assume that for any finite play prefix π we have $\bigcap\{\mathrm{act}(\mathrm{last}(\pi')) \mid \pi' \sim_1^* \pi\} \neq \emptyset$. We add a new position x of player i to the game which is distinguishable from all other positions for both players. Now if u is a position, $a \in \bigcup\{\mathrm{act}(w) \mid w \in [u]_{\sim_i}\}$ and $v \in [u]_{\sim_1}$ with $a \notin \mathrm{act}(v)$, then we define $f_a(v) = x$. Furthermore we define $\mathrm{act}(x) = x$ for some new action x and $f_x(x) = x$. Now any play that reaches x (and from then on clearly remains in x) is won by player $1 - i$. Using condition (C2) and our additional assumption one can show that if we have done this construction for $i = 0, 1$ then a player $i \in \{0, 1\}$ has a winning strategy from a position v_0 in the new game if and only if he has a winning strategy from v_0 in the original game.

Notice that however, the construction may change the winning condition. For example a reachability condition is not necessarily transformed into a reachability condition. A possibility to overcome this problem would be to allow such positions at which a player has lost immediately, independently of the winning condition in our model in general. The usual way is to allow terminal positions in the game and to say that a player has lost, if a terminal position is reached where he has to move.

Even without condition (C3), we could of course obtain a model, where in each play, starting from some position in a set $V' \subseteq V$, player $i \in \{0,1\}$ always knows which actions are available to him when it is his turn. This can be enforced by the following condition. Notice that for $V = V'$ this condition is equivalent to (C3).

(C3') If $\pi, \pi' \in P_{\text{fin}}(V')$ with $\pi \sim_i^* \pi'$ and $\text{last}(\pi), \text{last}(\pi') \in V_i$ then $\text{act}(\text{last}(\pi)) = \text{act}(\text{last}(\pi'))$.

Strategies. We have required that a strategy for player i is defined on all finite play prefixes such that at the last position it is his turn. Of course, the value of a strategy on play prefixes which are not compatible with the strategy is not relevant for the characteristic of being a winning strategy. So we could say that a strategy for player i from positions in $V' \subseteq V$ is a function $f : \text{dom}(f) \subseteq \{\pi \in P_{\text{fin}}(V') \mid \text{last}(\pi) \in V_i\} \to A$, such that the following conditions hold.

(1) $v \in \text{dom}(f)$ for all $v \in V' \cap V_i$.
(2) If $\pi \in P_{\text{fin}}(V')$ with $\text{last}(\pi) \in V_i$ is compatible with f, then $\pi \in \text{dom}(f)$.
(3) If $\pi, \pi' \in \text{dom}(f)$ such that $\pi \sim_i^* \pi'$ then $f(\pi) = f(\pi')$.

Now as long as condition (C2) is assumed and we furthermore require that for any $\pi \in P_{\text{fin}}(V')$ we have $\bigcap\{\text{act}(\text{last}(\pi')) \mid \pi' \sim_1^* \pi\} \neq \emptyset$, then this definition does not affect the existence of winning strategies.

Another requirement in our model is that a player cannot choose actions of which he does not know that they are available to him. Of course we can skip this condition, but we have to say what happens if a player chooses an action which is not available at the recent position. One possibility would be to say that the player loses in that case. (This is exactly what we have done when we have introduced condition (C3) to an arbitrary game.) Or the player could be informed about the fact that the action that he has chosen is not available at the recent position and then he may choose another action. This is also a special case of our model.

Perfect Recall and Number of Moves. In Section 2.2 we have introduced perfect recall and synchronous multiagent systems. Clearly the relation $\sim_i^*$ always fulfills both conditions. The relation $\sim_i^+$ fulfills only the first condition in general, because if private moves are hidden a player does not necessarily observe all the moves that are performed and so in particular he does not necessarily know the number of moves that have been performed up to a certain point.

Connections Between $\sim_i^V$ and $\sim_1^A$. We now introduce two further constraints that may be put on the epistemic components of games with partial information. These constraints concern the interaction between the indistinguishability of actions and positions.

- We say that $\sim_i^A$ implies $\sim_i^V$, if for all $u, v \in V$ with $u \sim_i^V v$ and all $a \in \text{act}(u)$ and $b \in \text{act}(v)$ with $a \sim_i^A b$ we have $f_a(u) \sim_i^V f_b(v')$.
- We say that $\not\sim_i^A$ implies $\not\sim_i^V$, if for all $u, v \in V$ with $u \sim_i^V v$ and all $a \in \text{act}(u)$ and $b \in \text{act}(v)$ with $a \not\sim_i^A b$ we have $f_a(u) \not\sim_i^V f_b(v)$.

Notice that even if $\not\sim_i^A$ does not imply $\not\sim_i^V$, after the action a has been executed, player i knows that the recent position is some position which is reachable via the execution of some action that he cannot distinguish from a at some position that he cannot distinguish from u. But if $f_b(u)$ is such a position, then player i might consider the position $f_b(u)$ possible which wouldn't be the case, if $\not\sim_1^A$ would imply $\not\sim_1^V$.

2.6 Winning Conditions

For a game $G = (V, V_0, (f_a)_{a\in A}, W_0)$, the winning condition $W_0 \subseteq V(AV)^\omega$ is not restricted in any way. But of course we want to have classes of games with interesting properties, like for example positional determinedness, finite representability and good algorithmic properties. So we have to consider certain restricted classes of winning conditions, which yield such properties while still being powerful enough to apply to many modelling problems.

We introduce such a class in the next section. We are particularly interested in winning condition which depend only on the positions that occur in a play and not on the actions. We call such winning conditions position based.

We define position based winning conditions and the corresponding notion of position based strategies and we will see that for games with full information and position based winning conditions, position based strategies suffice to win. On the other hand, for games with partial information this is not true in general, even if the winning condition is observation based. We will see this by an example after we have introduced observation based winning conditions.

Definition 2.11. Let $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a game and let $i \in \{0, 1\}$. For a sequence $\pi = v_0a_0v_1 \ldots \in V(AV)^* \cup V(AV)^\omega$ we write π_V for the sequence $v_0v_1 \ldots$ of positions in π.

The winning condition W_i for player i is called *position based* if for all $\pi, \pi' \in V(AV)^\omega$ with $\pi_V = \pi'_V$ we have $\pi \in W_i$ if and only if $\pi' \in W_i$. Clearly, if W_i is position based for $i \in \{0, 1\}$ then W_{1-i} is position based as well. We denote $V^\omega(W_i) = \{\pi_V \mid \pi \in W_i\}$.

A strategy $g : \{\pi \in P_{\text{fin}}(V') \mid \text{last}(\pi) \in V_i\} \to A$ for player i for G from positions in $V' \subseteq V$ is called *position based*, if for all finite play prefixes $\pi, \pi' \in \text{dom}(g)$ with $\pi_V = \pi'_V$ we have $g(\pi) = g(\pi')$.

Proposition 2.2. *Let $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a game such that W_0 is position based, let $v \in V$ and let $i \in \{0, 1\}$. If player i has a winning strategy for G from v, then he has a position based winning strategy for G from v.*

Proof. Let $f : \{\pi \in P_{\text{fin}}(v) \mid \text{last}(\pi) \in V_i\} \to A$ be a winning strategy for player i for G from v and let $V(P_{\text{fin}}(v)) = \{\pi_V \mid \pi \in P_{\text{fin}}(v)\}$. First there is a function $\zeta : V(P_{\text{fin}}(v)) \to A^*$ such that the following holds. If $v_0 \ldots v_n \in V(P_{\text{fin}}(v))$ and $\zeta(v_0 \ldots v_n) = a_0 \ldots a_{n-1}$, then we have $v_0a_0 \ldots a_{n-1}v_n \in P_{\text{fin}}(v)$ and $\zeta(v_0 \ldots v_{n-1}) = a_0 \ldots a_{n-2}$. Furthermore, if $(v_{n-1}, v_n) \in E_a$ for $a = f(v_0a_0 \ldots a_{n-2}v_{n-1})$ then $a_{n-1} = a$.

We define the strategy $g : \{\pi \in P_{\text{fin}}(v) \mid \text{last}(\pi) \in V_i\} \to A$ as follows. For $\pi = v_0 a_0 \dots a_{n-1} v_n \in \text{dom}(g)$ let $a'_0 \dots a'_{n-1} \in A^*$ with $\zeta(v_0 \dots v_n) = a'_0 \dots a'_{n-1}$ and let $g(\pi) := f(v_0 a'_0 \dots a'_{n-1} v_n)$. Obviously g is position based.

Now let $\pi = v_0 a_0 v_1 \dots \in P(v)$ be an arbitrary play of G from v_0 that is compatible with g and let $(a'_n)_{n<\omega}$ be defined by $a'_0 \dots a'_n := \zeta(v_0 \dots v_{n+1})$ for $n < \omega$. (Notice that by definition the function ζ is compatible with prefixes.) Then we have $\pi_n = v_0 a'_0 v_1 \dots a'_{n-1} v_n \in P_{\text{fin}}(v)$ for all $n < \omega$ and $\pi' = v_0 a'_0 v_1 a'_1 \dots \in P(v)$.

If furthermore, $n < \omega$ such that $v_n \in V_i$, then since π is compatible with g, the definition of g yields $f(\pi_n) = g(v_0 a_0 \dots a_{n-1} v_n) = a_n$ and thus $(v_n, v_{n+1}) \in E_{f(\pi_n)}$. So by definition of ζ we have $a'_n = f(\pi_n)$ and thus, π' is compatible with f and therefore won by player i. Since the sequence of positions of π coincides with the sequence of positions of π' and W_0 is position based, π is won by player i. Thus, g is a position based winning strategy for player i for G from v. ☐

Now we define partial information winning conditions and observation based winning conditions and we compare these two notions. Furthermore we will see that for observation based winning conditions, position based winning strategies do *not* suffice to win.

Definition 2.12. Let $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game, let $i \in \{0, 1\}$ and let $\sim'_i$ be an equivalence relation on the set $P(V)$. The winning condition W_i is called a *partial information winning condition* with respect to $\sim'_i$, if for all $\pi, \pi' \in P(V)$ with $\pi \sim'_i \pi'$ we have $\pi \in W_i$ if and only if $\pi' \in W_i$.

Now let $\mathcal{G} = (G, (\text{vis}_i^V), (\text{vis}_i^A))$ be a partial information game. We define equivalence relations $\sim_i^\omega$ and $\sim_i^{+,\omega}$ on $P(V)$ via the equivalence relations on positions and actions just as for finite play prefixes.

(ω) $\pi \sim_i^\omega \pi'$ if and only if $\text{first}(\pi) \sim_i^V \text{first}(\pi')$ and $\pi(j) \sim_i \pi'(j)$ for all $j < \omega$.

Using the indistinguishabilities of positions of the game graph we can introduce partial information position based winning conditions which we call observation based.

Definition 2.13. Let $\mathcal{G} = (G, (\text{vis}_i^V), (\text{vis}_i^A))$ be a game with partial information and let $i \in \{0, 1\}$. The winning condition W_i of player i is called *observation based*, if for all $\pi = v_0 a_0 v_1 \dots, \pi' = w_0 b_0 w_1 \dots \in V(AV)^\omega$ with $v_j \sim_i^V w_j$ for all $j < \omega$ we have $\pi \in W_i$ if and only if $\pi' \in W_i$.

Clearly, an observation based winning condition is a partial information winning condition with respect to $\sim_i^\omega$ and it is position based. However, the converse is not true in general, that means, a partial information winning condition with respect to $\sim_i^\omega$ which is position based is not necessarily observation based.

Now we consider an example of a game $\mathcal{G}$ with partial information and a very simple observation based winning condition W_1 such that player 1 has a winning strategy for $\mathcal{G}$ from a position v but he does not have a position based winning strategy from v.

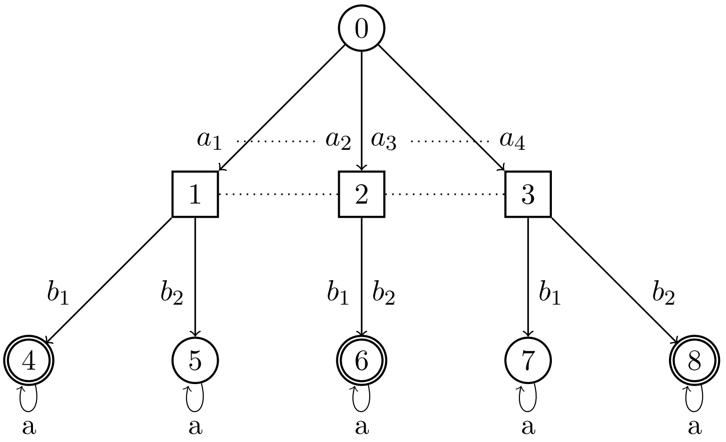

Figure 2.2: The game $\mathcal{G}$.

Example 2.2. *Consider the game* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$*,* $G = (V, V_0, (f_a)_{a\in A}, W_0)$ *as depicted in Figure 2.2. The dotted lines define the information of player* 1 *and player* 1 *wins a play of the game if one of the positions* 4*,* 6 *and* 8 *is reached. Finally, circle position belong to player* 0*. For trivial reasons, the winning condition* W_1 *is observation based.*

Clearly the strategy f *with* $f(0a_22) = f(0a_11) = b_1$ *and* $f(0a_32) = f(0a_43) = b_2$ *is a winning strategy for player* 1 *for* $\mathcal{G}$ *from* 0*. But player* 1 *does not have a position based winning strategy for* $\mathcal{G}$ *from* 0 *since each such strategy has to yield the same value on* $0a_22$ *and on* $0a_32$ *and thus it also has to yield the same value on* $0a_11$ *and on* $0a_43$*.*

2.6.1 Muller-Games

Muller-games form a widely considered subclass of Borel-games in Computer Science. We do not introduce Borel-sets formally, see for example [Mos80]. Nevertheless we define the notion of a Borel-game and we note the following result by Martin which has been proved in [Mar75]. It is one of the deepest result in the theory of infinite two-player win-loss games. Although there are nondetermined games with full information, Martin's Theorem says that one has to consider complicated winning conditions to find one. In the very contrary, games with partial information are not determined, even for very simple winning conditions. We will see an example at the end of this section.

Definition 2.14. A game $G = (V, V_0, (f_a)_{a\in A}, W_0)$ is called a *Borel-game* if the winning condition W_0 is position based and $V^\omega(W_0)$ is a Borel-subset of V^ω. Clearly, if $V^\omega(W_0)$ is a Borel-subset of V^ω then $V^\omega(W_1) = V^\omega \setminus V^\omega(W_0)$ is a Borel-subset of V^ω as well.

Theorem 2.1. (Martin) *Each Borel-game is determined.*

Now we introduce Muller-games with finitely many colors and we consider certain special cases of Muller-games, especially parity games.

Definition 2.15. Let $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game. A *coloring* of G is a function $\mathrm{col} : V \to C$ into some set $C \subseteq \omega$. The coloring is called finite, if C is finite, that means, $C \subsetneq \omega$. If $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ is a game with partial information, then a coloring col of G is called *compatible with the information of player* $i \in \{0, 1\}$, if $\mathrm{col}(u) = \mathrm{col}(v)$ for all $u, v \in V$ with $u \sim_i^V v$.

Definition 2.16. A *Muller-game* is a game $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0))$ where $\mathrm{col} : V \to C$ is a finite coloring of G and $\mathcal{F}_0 \subseteq 2^C$ is a set. The winning condition W_0 of G is defined as follows. For each $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in V(AV)^\omega$ we have $\pi \in W_0$ if and only if $\inf_{\mathrm{col}}(\pi) = \{c \in C \mid \mathrm{col}(v_i) = c \text{ for infinitely many } i < \omega \} \in \mathcal{F}_0$. In particular, W_0 is position based.

Definition 2.17. A *parity game* is a Muller-game where the component $\mathcal{F}_0$ is given by $\mathcal{F}_0 = \{C' \subseteq C \mid \min(C') \text{ is even } \}$.

So player 0 wins a play of a parity game, if the least color which is seen infinitely often is even. Since for a parity game $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0))$ the component $\mathcal{F}_0$ is fixed, we denote the game as $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$.

Parity games are a special case of Borel games, so by Theorem 2.1, parity games are determined. Moreover they are positional determined, which has first been shown by Emerson and Jutla in [EJ91]. (The result has been found independently by Mostowski, cf. [Mos91].) Jurdziński developed a couple of quite fast algorithms for solving the strategy problem for parity games (and constructing corresponding winning strategies), cf. [Jur00]. We can summarize those results as follows.

Theorem 2.2. *Parity games are uniformly positional determined and for finite parity games, the winning regions as well as corresponding uniform winning strategies can be constructed in time* $|V|^{O(|C|)}$.

The complexity bound is a quite rough estimation, but for our concerns this one does suffice. The important point here is that this bound is only exponential in the number of colors and not in the number of positions. Now from this result we obtain the following result for Muller-games, using the LAR-reduction of Muller-games to parity games. LAR is the shortcut for latest appearance record. Using the LAR-reduction of Muller-games to parity games it can be shown that each Muller-game is determined with LAR-memory. Since we do not need the internal structure of the memory, we just estimate its size. The idea of game reduction has first been used by Thomas in [Tho95]. We do not carry out the construction here, see for example [GTW02].

Theorem 2.3. *Muller-games are uniformly determined with memory* $(|C|)!$ *and for finite Muller-games, the winning regions as well as corresponding uniform winning strategies can be constructed in time* $|V|^{|C|^k}$ *for some* $k < \omega$.

Definition 2.18. A game $G = (V, V_0, (f_a)_{a \in A}, R)$ with $R \subseteq V$ is called a *Büchi-game*, if the winning condition W_0 of G is defined as follows. For each sequence $\pi = v_0 a_0 v_1 a_1 \ldots \in V(AV)^\omega$ we have $\pi \in W_0$ if and only if $\inf(\pi) := \{v \in V \mid v_i = v \text{ for infinitely many } i < \omega \} \cap R \neq \emptyset$. The game is called a *co-Büchi-game* if the winning condition is defined as follows. For each $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in V(AV)^\omega$ we have $\pi \in W_0$ if and only if $\inf(\pi) \subseteq R$.

Notice that a Büchi-game $G = (V, V_0, (f_a)_{a \in A}, R)$ is a parity game where the coloring function $\mathrm{col} : V \to \{0, 1\}$ is defined by $\mathrm{col}(R) := \{0\}$ and $\mathrm{col}(V \setminus R) := \{1\}$. So from Theorem 2.2 we immediately obtain the following result.

Theorem 2.4. *Büchi-games are uniformly positional determined and the strategy problem for finite Büchi-games is in P*

Definition 2.19. A game $G = (V, V_0, (f_a)_{a \in A}, R)$ with $R \subseteq V$ is called a *reachability game* if the winning condition W_0 of G is defined as follows. For each $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in V(AV)^\omega$ we have $\pi \in W_0$ if and only if $\mathrm{occ}(\pi) := \{v \in V \mid v_i = v \text{ for some } i < \omega\} \cap R \neq \emptyset$. The game is called a safety (or a co-reachability) game if the winning condition is defined as follows. For each $\pi = v_0 a_0 v_1 a_1 v_2 \ldots \in V(AV)^\omega$ we have $\pi \in W_0$ if and only if $\mathrm{occ}(\pi) \subseteq R$.

Reachability games are not special cases of Büchi-games in the direct sense, that means, in general we cannot just find a set $R' \subseteq V$ such that the corresponding Büchi-condition coincides with the given reachability condition. But there is a very simple transformation of reachability games into Büchi-games which preserves the existence of winning strategies for both players from all positions. For a reachability game $G = (V, V_0, (f_a)_{a \in A}, R)$, let $G' = (V, V_0, (f'_a)_{a \in A}, R)$ be the Büchi-game which is obtained from G as follows. For $u \in V \setminus R$ and $a \in \mathrm{act}(u)$ we define $f'_a(u) := f_a(u)$. For $u \in R$ and $a \in \mathrm{act}(u)$ we define $f'_a(u) := u$. Clearly this construction can be done in time linear in the size of the game graph. The same construction can of course be applied to transform each safety game into a co-Büchi game. There we have to change the edges at positions from $V \setminus R$ in the same way as we have done it here for positions from R. So from Theorem 2.4 we immediately obtain the following result.

Theorem 2.5. *(Co-) Reachability games are uniformly positional determined and the strategy problem for finite reachability games is in P.*

Now if $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ is a reachability game with partial information then we can ask whether for the game $\mathcal{G}' = (G', (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$, again, each player has a winning strategy for $\mathcal{G}'$ from some position $v \in V$ if and only if he has a winning strategy for $\mathcal{G}$ from v. To answer this positively we have to make an assumption about the actions in G. For instance, it is sufficient that for $i \in \{0, 1\}$ we have $\mathrm{act}(u) = \mathrm{act}(v)$ for all $u, v \in V_i$ with $u \sim_1^V v$. We have to require this to be able to make the winning strategy constant over equivalence classes of play prefixes, if we transport it from one game to the other in the obvious way.

Now we shall see that even for safety winning conditions, games with partial information are not determined in general. Furthermore we have a look at a safety game where player 1 has a winning strategy from a position v but he does not have a positional winning strategy from v. In both cases we use a winning condition which is compatible with the information of both players, that means, for $i \in \{0, 1\}$, for all u, v with $u \sim_i^V v$ we have $u \in R$ if and only if $v \in R$.

Example 2.3. *Consider the game $\mathcal{G}$ from Example 2.2. We change the information of player* 1 *about the actions by defining* $\mathrm{vis}_1^A(a) = \mathrm{vis}_1^A(a') = \mathrm{vis}_1^A(b) = \mathrm{vis}_1^A(b') = a$

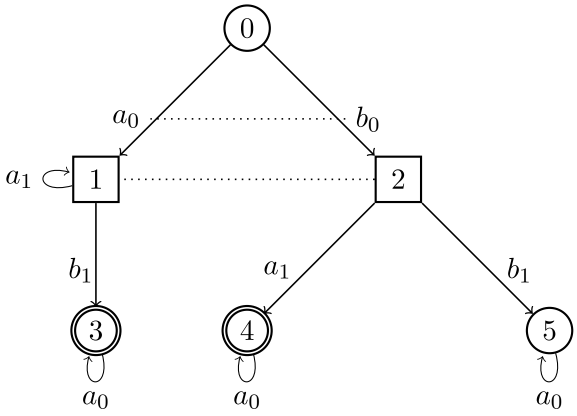

Figure 2.3: The game $\mathcal{G}$.

and we call the resulting game $\mathcal{G}'$. *Clearly, player* 0 *does not have a winning strategy for* $\mathcal{G}'$ *from initial position* 0. *But now, player* 1 *does not have a winning strategy from initial position* 0 *as well, because he has to choose the same action at* $0a'1$ *and* $0b'3$ *and so player* 0 *has a counter strategy for each strategy of player* 1. *Thus, the partial information safety game* $\mathcal{G}'$ *is not determined.*

Example 2.4. *Consider the game* $\mathcal{G}$ *as depicted in Figure 2.3. The dotted lines define the information of player* 1 *and player* 1 *wins a play of the game one of the positions* 3 *and* 4 *is reached. Finally, circle positions belong to player* 0.

Clearly the strategy f *with* $f(0b_02) = f(0a_01) := a_1$ *and* $f(0a_01a_11) := b_1$ *is a winning strategy for player* 1 *for* $\mathcal{G}$ *from initial position* 0. *But it is easy to see that player* 1 *does not have a positional winning strategy for* $\mathcal{G}$ *from initial position* 0.

Remark. Notice that in this example player 1 does not even have a winning strategy from 0 which is a positional strategy for G. The fact that memoryless winning strategies do not suffice to win in safety games with partial information can already be seen by the following much simpler example. Let $V = \{0, 1, 2, 3\}$, $V_0 = \emptyset$, $A = \{a, b\}$, $R = \{0, 1, 2\}$ and let the availability of actions be as follows. $0 \xrightarrow{a} 1$, $1 \xrightarrow{a} 2$, $1 \xrightarrow{b} 3$, $2 \xrightarrow{a} 2$ and $3 \xrightarrow{a} 3$. The epistemic component is defined via $0 \sim_1^V 1$. Clearly player 1 has a winning strategy for this game from 0 but he does not have a positional winning strategy from 0. Notice that the game graph of this game is a tree.

2.7 Nondeterministic Games

We consider nondeterministic games since the construction that we will use to turn a game with partial information into a game with full information yields a nondeterministic game in general.

The definition of a nondeterministic two-player zero-sum game is the obvious generalization of our definition of a game, so it is a tuple $G = (V, V_0, (E_a)_{a \in A}, W_0)$ with $E_a \subseteq V \times V$ for $a \in A$ and all the other components are as before. Plays, strategies, memory strategies and winning strategies are defined as before as well with the only difference that now there are also *finite* plays since we do not require the edge-relation $\bigcup\{E_a \,|\, a \in A\}$ to be serial. Player $i \in \{0,1\}$ wins a finite play if the last position of the play is a terminal position which belongs to player $1-i$. Of course a strategy has now to be defined only on finite play prefixes where the last position is a nonterminal one.

Clearly, nondeterministic games are not determined in general, even for very simple winning conditions. For example consider the nondeterministic reachability game $G = (\{0,1,2\}, \emptyset, E_a, \{1\})$ where we have only one player and only one action a and $E_a = \{(0,1),(0,2),(1,1),(2,2)\}$. In fact, none of the players has a real influence on the game. Player 1 just chooses the single action which is available at 0 but he cannot choose the next position. So neither can player 0 force the game into 1, nor can player 1 force the game into 2, so the game is not determined. Thus we cannot find a deterministic game $H = (V, V_0, (f_a)_{a\in A}, W_0)$ and an injection $f : \{0,1,2\} \to V$ such that for $i \in \{0,1\}$, for each $u \in \{0,1,2\}$ player i has a winning strategy for G from u if and only if he has a winning strategy for H from $f(u)$. So the observation is that in general, nondeterministic games are not equivalent to deterministic games.

Nevertheless, for each nondeterministic game $G = (V, V_0, (E_a)_{a\in A}, W_0)$ and each $i \in \{0,1\}$ we can construct a deterministic game $H = (V \cup V', V_0', (f_a)_{a\in A'}, W_0')$ such that for each $v \in V$, player i has a winning strategy for G from v if and only if he has a winning strategy for H from v. In the following, let T be the set of terminal positions in G and let $i \in \{0,1\}$. We assume that $A \cap V = \emptyset$. Now the player i determinization of G is the game

$$G^i := (V \uplus V', V_0', (f_a)_{a\in A}, (f_v)_{v\in V}, W_0')$$

with the following components.

- $V' = \{(v,a) \in V \times A \,|\, a \in \mathrm{act}(v)\}$.
- $V_0' = V_0 \setminus T$, if $i = 1$ and $V_0' = V_0 \cup T \uplus V'$, if $i = 0$.
- $f_a(v) := (v,a)$ for $(v,a) \in V'$.
- $\mathrm{dom}(f_v) = \{(w,a) \in V' \,|\, (w,v) \in E_a\}$, if $v \notin T$.
  $\mathrm{dom}(f_v) = \{(w,a) \in V' \,|\, (w,v) \in E_a\} \cup \{v\}$, if $v \in T$.
- $f_v(x) = v$ for all $x \in \mathrm{dom}(f_v)$.

Finally, for a play π of G^i we define the membership in W_0' as follows. If no position from V which occurs in π is a terminal position in the game G, then π is in W_0' if and only if the play in G, which is obtained from π by deleting all positions and actions in π, which do not belong to V and A, respectively, is in W_0. If a position from V which occurs in π is a terminal position in the game G, then π is in W_0 if and only if this position is in V_1.

Proposition 2.3. *For all $v \in V$, player $1-i$ has a winning strategy for G from v if and only if he has a winning strategy for G^i from v.*

A nondeterministic game $G = (V, V_0, (E_a)_{a\in A}, W_0)$ is called a Muller-game (a parity game, a Büchi-game, a reachability game) if the set W_0 is given by a Muller-condition $(\mathrm{col}, \mathcal{F}_0)$ (a parity condition, a Büchi-condition, a reachability condition). If G is a nondeterministic Muller-game with colors $0, \dots, d-1$ for some $0 < d < \omega$ and $i \in \{0, 1\}$, then the player i determinization of G is a Muller-game as well. The coloring col' can be defined by $col'(v) := \mathrm{col}(v)$ for all positions $v \in V \setminus T$, $col'(v, a) := \mathrm{col}(v)$ for all $(v, a) \in V'$, $col'(v) := d + 1$ for all $v \in T \cap V_1$ and $col'(v) := d + 2$ for all $v \in T \cap V_0$. The corresponding winning component is then $\mathcal{F}'_0 = \mathcal{F}_0 \cup \{\{d + 1\}\}$.

In the same way, the determinization of a nondeterministic game preserves parity conditions, Büchi-conditions and if there are no terminal positions of player 0, then it also preserves reachability conditions. (Analog, if there are no terminal positions of player 1, then the determinization preserves safety conditions.) For parity conditions we color the terminal positions of player 1 with 0 and the terminal positions of player 0 with 1. For Büchi-games and reachability games we add the terminal positions of player 1 to R.

Now clearly, for finite games this construction can be done in time linear in the size of the game graph. So in particular all the results about the complexity of solving the strategy problem for the different classes of Muller-games from Section 2.6.1 hold for nondeterministic games as well. Furthermore the results on the kind of strategies which are needed to win in the different classes of Muller-games from Section 2.6.1 also hold for nondeterministic games as well. For example, the strategy problem for finite nondeterministic parity games can be solved in time $|V|^{O(|C|)}$ and both players have uniform positional strategies on their winning regions. (However, notice that the union of the two winning regions now is not the set of all positions in general.)

2.8 Comparison with other Models

Reif-games. In [Rei84], Reif has suggested the following model. He has used games with partial information to define the semantics of so called private alternating Turing-machines. These machines are a generalization of alternating Turing-machines.

Definition 2.20. A *Reif-game* is a two-player game of the form $G = (V, E)$ where $V \subseteq \{0, 1\} \times P_0 \times C \times P_1$ for some sets P_0, C and P_1 is the set of positions and $E \subseteq V \times V$ is the move relation.

For a position $p = (i, p_0, c, p_1) \in V$ it is player i's turn. A play of G from initial position $v_0 \in V$ is a finite or infinite sequence $\pi = v_0 v_1 v_2 \ldots \in V^* \cup V^\omega$ such that $(v_j, v_{j+1}) \in E$ for all $j < |\pi|$ and $\mathrm{last}(\pi)E = \emptyset$, if π is finite. Player i wins a finite play π if $\mathrm{last}(\pi) \in V_{1-i}$. Each infinite play is a draw.

These are the basic notions concerning the physical part of the model. Notice that in Reif-games, the players move by choosing a next position and not by choosing an action. So for Reif-games we need a new notion of strategies. We shall define it after we have introduced the epistemic component of a Reif-game. This component is already implicit in the physical part.

For a position (j, p_0, c, p_1) we say that p_0 is the private state of player 0, p_1 is the private state of player 1 and c is the common state. Intuitively, a player can see his own private state, the common state and whose turn it is, but he cannot see the private state of the other player. So for $i \in \{0, 1\}$ we define $\mathrm{vis}_i(j, p_0, c, p_1) = (j, p_i, c)$ and $\mathrm{priv}_i(j, p_0, c, p_1) = p_i$ for all $(j, p_0, c, p_1) \in V$. Reif has required the following two conditions for $i = 0, 1$.

(R1) If $v \in V_i$ and $(v, v') \in E$ then $\mathrm{priv}_{1-i}(v) = \mathrm{priv}_{1-i}(v')$.
(R2) If $v, w \in V_i \setminus T$ and $v \sim_i w$ then $\{\mathrm{vis}_i(v') \mid (v, v') \in E\} = \{\mathrm{vis}_i(w') \mid (w, w') \in E\}$.

Where T is the set of terminal positions in G.

Condition (R1) says that a player cannot modify the private state of the other player. Condition (R2) essentially says that when a player has to move then he knows which moves he can make (up to terminal positions). Because, according to condition (R1), what happens in a move of a Reif-game is that the player who moves changes (at most) the part of the position that is visible to him. This change then determines the next position, conditional on the private state of the other player which the player whose turn it is can neither see nor change. This also shows that we have to define actions in Reif-games by labelling an edge $(v, w) \in E$ with $v \in V_i$ by $\mathrm{vis}_i(w)$.

Now it is clear how strategies in Reif-games should be defined and this is in fact exactly the way how Reif has done it. The two equivalence relations on finite play prefixes are defined as we have done this for our model via different iterations of the function vis_i. We only define strategies for Reif-games. Strategies if private moves are hidden are defined analog.

Definition 2.21. Let $G = (V, E)$ be a Reif-game and let $V' \subseteq V$. A *strategy* for player i for G from initial positions in V' is a function $f : \{\pi \in P_{\mathrm{fin}}(V') \mid \mathrm{last}(\pi) \in V_i \setminus T\} \to V$ such that the following two conditions hold.

(1) For all $\pi \in \mathrm{dom}(f)$ we have $(\mathrm{last}(\pi), f(\pi)) \in E$.
(2) For $\pi, \pi' \in \mathrm{dom}(f)$ with $\pi \sim_i^* \pi'$ we have $f(\pi) \sim_i f(\pi')$.

Now let f be a strategy for player i for G from initial positions in V'. A prefix $\pi = v_0 v_1 v_2 \ldots$ of a play in G from initial position $v_0 \in V'$ is called compatible with f if for all $j < |\pi|$ such that $v_j \in V_i$ we have $v_{j+1} = f(v_0 \ldots v_j)$.

Now how can we turn a Reif-game into a game with partial information in a reasonable way and which special properties do the resulting games have? We have introduced actions to the model and clearly these actions are deterministic. Furthermore, the visibilities of positions are already defined and they also yield the visibilities of the actions in the obvious way. But now there is an essential difference between our model and Reif-games. In Reif-games there are finite plays and each infinite play is a draw. In our model there are only infinite plays and no play is a draw.

To level this difference we proceed as follows. First we introduce a new action $\circlearrowleft$ to a Reif-game $G = (V, E)$ and we let $\mathrm{dom}(f_\circlearrowleft)$ be the set of all terminal positions in G. Now for each $v \in \mathrm{dom}(f_\circlearrowleft)$ we define $f_\circlearrowleft(v) := v$. To define the winning condition we have two possibilities. We can make a reachability game out of G or we can make a safety game out of G. For the reachability case we define $R = \mathrm{dom}(f_{\circlearrowleft}) \cap V_1$ and for the safety case we let $R = V \setminus (\mathrm{dom}(f_{\circlearrowleft}) \cap V_0)$. We call $R(G)$ the corresponding reachability game with partial information for G and we call $S(G)$ the corresponding safety game with partial information for G. Then for each $v \in V$, Player 0 has a winning strategy for G from v if and only if he has a winning strategy for $R(G)$ from v and player 1 has a winning strategy for G from v if and only if he has a winning strategy for $S(G)$ from v.

Now we list some important properties of the games that arise from such transformations of Reif-games. Some of them are inherent to the game model and some are due to the conditions (R1) and (R2).

Proposition 2.4. *Let* $i \in \{0, 1\}$.

(1) $\sim_i^A$ *implies* $\sim_i^V$ *and* $\not\sim_i^A$ *implies* $\not\sim_i^V$.
(2) If $u, v \in V_i \setminus T$ *with* $u \sim_i^V v$, *then* $\mathrm{act}(u) = \mathrm{act}(v)$.
(3) For all $v \in V$ *we have* $\{u \in V \mid u \sim_0^V v\} \cap \{u \in V \mid u \sim_1^V v\} = \{v\}$.
(That means, the recent position is distributed knowledge.)
(4) If $u, v \in V_i$ *with* $u \sim_i^V v$ *and* $u \neq v$ *then* $uE \cap vE = \emptyset$.
(5) If $u \in V_i$ *then* $v \not\sim_i^V u$ *for all* $v \in uE \setminus \{u\}$.

Game Structures of Incomplete Information. In [dWDHR06], games with partial information have been used to solve the universality problem of finite automata. Game structures of incomplete information, which have been suggested in [CDHR06], are suited for this problem. The definition is slightly adapted to make it more compatible with our notation. In [CDHR06] only finite games have been considered. But since there are no intrinsic reasons to restrict the definition to finite games, we do not do so.

Definition 2.22. A *game structure of incomplete information* is given by a tuple $G = (V, v_0, (E_a)_{a \in A}, \gamma)$, where V is a set of positions, $v_0 \in V$ is the initial position, $A \neq \emptyset$ is the set of actions, $E_a \subseteq V \times V$ is a binary relation on V for each $a \in A$ and $\gamma : \mathrm{Obs} \to 2^V \setminus \emptyset$ is a function for some set Obs, such that the following two conditions hold.

(1) For all $a \in A$ the relation E_a is serial.
(2) $\{\gamma(o) \mid o \in \mathrm{Obs}\}$ is a partition of V.

A play of G is a sequence $v_0 a_0 v_1 \ldots \in V(AV)^\omega$ with $(v_i, v_{i+1}) \in E_{a_i}$ for all $i < \omega$.

Notice that in this model there is only *one* partition $\{\gamma(o) \mid o \in \mathrm{Obs}\}$ of the set of positions which defines the information of player 1 about the positions of the game. Player 0 has full information. Furthermore notice that actions are nondeterministic in this model and that there is no partition of the set of positions into positions of player 0 and player 1. The reason for this is that the positions of player 0 are implicit in the model. At a position $v \in V$, player 1 chooses an action $a \in A$ and player 0 chooses some edge $(v, w) \in E_a$. This determines the next position w of the game. So of course, strategies for player 0 are different objects than strategies for player 1. Finally, the initial position is already determined by the model itself and at each position, all actions are available.

The indistinguishability of play prefixes (for player 1) is defined via the *-iteration of the function vis_1 which is here given by $\mathrm{vis}_1(va) = oa$ for the uniquely determined $o \in \mathrm{Obs}$ such that $v \in \gamma(o)$. Now it is clear how strategies have to be defined in those games.

Definition 2.23. Let $G = (V, v_0, (E_a)_{a\in A}, \gamma)$ be a game structure of incomplete information. A *deterministic strategy for player* 1 for G is a function $f : P_{\text{fin}}(v_0) \to A$. The strategy is called *observation based*, if for all $\pi, \pi' \in P_{\text{fin}}(v_0)$ with $\pi \sim_1^* \pi'$ we have $f(\pi) = f(\pi')$. A *deterministic strategy for player* 0 for G is a function $f : P_{\text{fin}}(v_0)A \to V$ such that for all $\pi \in P_{\text{fin}}(v_0)$ and all $a \in A$, $(\mathrm{last}(\pi), f(\pi a)) \in E_a$.

Now let $G = (V, v_0, (E_a)_{a\in A}, \gamma)$ be a game structure of incomplete information and let $W_0 \subseteq V(AV)^\omega$ be a winning condition for the game. How can we turn G into a game with partial information in a reasonable way?

Let $G^0 := (V \uplus V', V_0 \cup V' \cup T, (f_a)_{a\in A}, (f_v)_{v\in V}, W_0')$ be the player 0 determinization of G (cf. Section 2.7) and let $\mathrm{vis}_0^V(v) = v$ for all $v \in V \cup V'$ and $\mathrm{vis}_0^A(a) = a$ for all $a \in A$. Furthermore, for each $v \in V$ and each $a \in \mathrm{act}(v)$ we let $\mathrm{vis}_1^V(v) := o$ and $\mathrm{vis}_1^V(v, a) = (o, a)$ for the uniquely determined $o \in \mathrm{Obs}$ with $v \in \gamma(o)$. For $a \in A$ we let $\mathrm{vis}_1^A(a) := a$ and for $v \in V$ we let $\mathrm{vis}_1^A(v) = 0$ where we assume that $0 \notin A$. So we have $\mathrm{VIS}_1^V = \mathrm{Obs} \uplus \mathrm{Obs} \times A$ and $\mathrm{VIS}_1^A = A \uplus \{0\}$.

Finally we denote $\mathcal{G} := (G^0, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ and we call $\mathcal{G}$ the corresponding game with partial information for G. Intuitively it is clear that $\mathcal{G}$ is indeed a reasonable match for the game G in the world of partial information games. In particular, for all finite play prefixes $\pi = v_0a_0 \dots a_{i-1}v_i \in P_{\text{fin}}(v_0)$ and $\pi' = w_0b_0 \dots b_{j-1}w_j \in P_{\text{fin}}(v_0)$ of G we have $\pi \sim_1^* \pi'$ if and only if $\overline{\pi} \sim_1^* \overline{\pi}'$ where $\overline{\pi}$ and $\overline{\pi}'$ are the corresponding play prefixes of G^0, that means, $\overline{\pi} = v_0a_0(v_0, a_0)v_1v_1(v_1, a_1)v_2v_2 \dots$ and $\overline{\pi}'$ analog. Using this it can easily be shown that player 1 has a deterministic observation based winning strategy for G if and only if he has a winning strategy for $\mathcal{G}$ from initial position v_0.

Now which special properties does such a game $\mathcal{G}$ as we have defined it from a game structure G of incomplete information have? The following proposition yields a complete list of all those properties. We will see afterwards what 'complete' means.

Proposition 2.5. *Let G be a game structure of incomplete information and let $\mathcal{G} = (G^0, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ be the corresponding game with partial information.*

(1) $\mathrm{dom}(f_a) = V_1$ for all $a \in A_1$ and vis_1^A is constant over A_0.
(2) For $i \in \{0, 1\}$, for all $v \in V_i$ and all $a \in \mathrm{act}(v)$ we have $f_a(v) \in V_{1-i}$.
(3) If $v, w \in V_1$ with $v \sim_1^V w$ then $f_a(v) \sim_1^V f_a(w)$ for all $a \in A_1$.
(4) Player 0 has full information.

Now let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a game with partial information such that the propositions (1) - (4) hold. As a technical simplification we furthermore assume that $A_0 \cap A_1 = \emptyset$. Then we can construct a game structure of incomplete information which is 'equivalent' to $\mathcal{G}$ in a similar sense as for the above construction. (And this is exactly what we mean by saying that (1) -

(4) are a 'complete' list.) Notice that we have to commit ourselves to a fixed initial position v_0 and we have to choose this position from the positions of player 1 since we shall eliminate all the positions of player 0 from the game. But we do not see this as a restriction of the model.

We define the game structure $H = (V_1, v_0, (E_a)_{a \in A_1}, \gamma)$ of incomplete information as follows. First let v_0 be an arbitrary position from V_1 and let γ be the identity on $V/\sim_1^V$. Now for $a \in A_1$ we define $E_a{=}\{(u,v){\in}V_1{\times}V_1 \mid \exists\, b{\in}A_0\ :\ f_b(f_a(u)){=}v\}$

Now again for all $\pi = v_0a_0 \dots a_{i-1}v_i, \pi' = w_0b_0 \dots b_{j-1}w_j \in P_{\text{fin}}(v_0)$ we have $\pi \sim_1^* \pi'$ if and only if $\overline{\pi} \sim_1^* \overline{\pi}'$ where $\overline{\pi}$ and $\overline{\pi}'$ are the corresponding prefixes in the game H, that means, $\overline{\pi} = v_0a_0v_2a_2v_4\dots$ and $\overline{\pi}'$ analog. So again it can be shown that player 1 has a winning strategy for $\mathcal{G}$ from v_0 if and only if he has a deterministic observation based winning strategy for H.

Chapter 3

Winning Strategies

In this chapter we study solutions for the strategy problem and the implementation of winning strategies in games with partial information. We are particularly interested in several special cases of Muller-conditions. First we consider the case where private moves are not hidden and in Section 3.4 we shall see how we can extend the results to the case where private moves are hidden. We show that the strategy problem for finite parity games with partial information is in EXPTIME and we prove upper and lower bounds on the memory that is needed to win in certain classes of games with partial information. Furthermore we show that even for Reif-games, the strategy problem is EXPTIME-hard. We also study the relationship between games and finite automata on infinite trees and we will see that there is an intimate connection between games with partial information and universal tree automata. Finally we present an optimized procedure for the evaluation of μ-calculus formulas on games which result from games with partial information via the powerset construction from Section 3.1.

3.1 Powerset Construction

First we consider a powerset construction which turns a game with partial information with an arbitrary winning condition into a nondeterministic game with full information such that the existence of winning strategies for at least one player is preserved. The construction has originally been suggested by John H. Reif in [Rei84]. Reif has considered safety objectives on a somewhat restricted game model (where the game graphs are finitely branching). The construction has been applied to another model (where the game graphs are finite) and arbitrary observation based winning conditions in [CDHR06]. We apply this construction to our model where the game graphs can be infinite branching and arbitrary winning conditions are allowed.

The idea of the construction is the following. Since the resulting game is supposed to have full information, both players always know the recent position of the game. So if the existence of a winning strategy for player i from initial position v_0 shall be preserved by the construction, then any position of the new game must capture all the uncertainties about the recent position that player i actually has after some finite prefix $\pi \in P_{\text{fin}}(v_0)$ has been played. That means, the position must contain all the positions, that player i considers possible after π has been played. So the positions of the new game are of the form

$$v(\pi) = \{\text{last}(\pi') \mid \pi' \sim_i^* \} \qquad \text{for } \pi \in P_{\text{fin}}(v_0).$$

Now let $\mathcal{G} = (G, (\text{vis}_i^V), (\text{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information, let $v_0 \in V$ and let $i \in \{0,1\}$. We define the corresponding game

$$\overline{G}_{v_0}^i = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A}, \overline{W}_0)$$

with full information as follows. To simplify the notation we let $i = 1$ and we denote the game $\overline{G}_{v_0}^i$ by $\overline{G}_{v_0}$.

- $\overline{V} = \{v(\pi) \mid \pi \in P_{\text{fin}}(v_0)\}$.
- $\overline{V}_0 = \{v(\pi) \in \overline{V} \mid \text{last}(\pi) \in V_0\}$.
- For $a \in A$, the edge relation $\overline{E}_a$ is the union of
  $\overline{E}_a^0 := \{(v(\pi), v(\pi b v)) \mid \ \pi, \pi b v \in P_{\text{fin}}(v_0),\ b \sim_1^A a,\ v(\pi) \in \overline{V}_0 \ \}$ and
  $\overline{E}_a^1 := \{(v(\pi), v(\pi a v)) \mid \ \pi, \pi a v \in P_{\text{fin}}(v_0),\ a \in \bigcap_{v \in v(\pi)} \text{act}(v),\ v(\pi) \in \overline{V}_1\}$.
- For a play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ in $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$ we define $\overline{\pi} \in \overline{W}_1$
  $:\Longleftrightarrow$
  for each play $\pi = v_0 a_0' v_1 a_1' \ldots \in P(v_0)$ in G from v_0 with $a_i' \sim_1^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$ we have $\pi \in W_1$.

Notice that the definition of $\overline{E}_a$ for $a \in A$ is not independent of the chosen π. To determine all a-successors of a position $v(\pi)$ for some $\pi \in P_{\text{fin}}(v_0)$ we have to look at all $\pi' \in P_{\text{fin}}(v_0)$ such that $v(\pi) = v(\pi')$. However, Proposition 3.1 tells us that it suffices to look at all $\pi' \in P_{\text{fin}}(v_0)$ such that $\pi' \sim_1^* \pi$, although in general we can have $v(\pi) = v(\pi')$ for prefixes $\pi, \pi' \in P_{\text{fin}}(v_0)$ with $\pi \not\sim_1^* \pi'$.

That we have to require $a \in \bigcap\{\text{act}(v) \mid v \in v(\pi)\}$ if $\text{last}(\pi) \in V_1$ is due the fact that player 1 is not allowed to choose actions of which he does not know that they are available to him. So in the corresponding game with full information, the respective edges have to be eliminated.

Remark. If $b \sim_1^A a$ then of course $\overline{E}_b^0 = \overline{E}_a^0$ and we have at most $a \in A_1$ or $b \in A_1$ but not both, and $\overline{E}_b^1 = \emptyset$ if $b \notin A_1$. Furthermore for all plays $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots,\ \overline{\pi}' = \overline{v}_0 a_0' \overline{v}_1 a_1' \ldots \in P(\overline{v}_0)$ with $a_i \sim_1^A a_i'$ for all $i < \omega$ we have $\overline{\pi} \in W_0$ if and only if $\overline{\pi}' \in W_0$. So if we take an arbitrary representative system $\overline{A}$ for $A/{\sim_1^A}$ with $A_1 \subseteq \overline{A}$ then the game $(\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{W}_0 \cap \overline{V}(\overline{A}\,\overline{V})^\omega)$ is equivalent to $\overline{G}_{v_0}$ and the game graph of this game coincides with the game graph of $\overline{G}_{v_0}$, independently of the choice of $\overline{A}$. The only thing that a variation of the representative system causes is a renaming of the labels on the edges from positions of player 0. So from now on we fix such a representative system $\overline{A}$ and we let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{W}_0)$ where we denote $\overline{W}_0 \cap \overline{V}(\overline{AV})^\omega$ by $\overline{W}_0$.

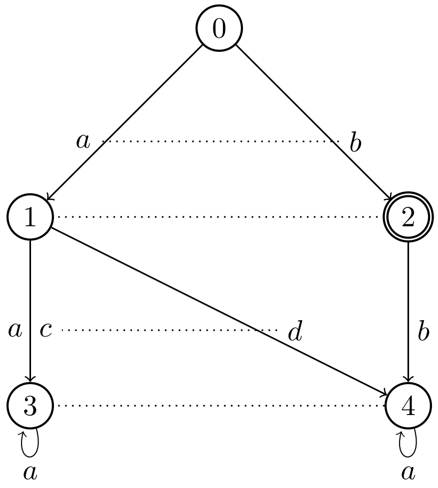

Figure 3.1: The game $\mathcal{G}$.

The next example shows that the winning condition $\overline{W}_1$ is not necessarily position based, even if W_1 is. However, as we will see in Section 3.1.1, $\overline{W}_1$ is position based, if W_1 is observation based and the game graph of G is finitely braching.

Example 3.1. *Consider the game $\mathcal{G}$ as depicted in figure 3.1. The dotted lines define the information of player* 1 *and player* 0 *wins a play of the game if position* 2 *is reached. Finally, circle positions belong to player* 0*. The corresponding game $\overline{G}_{v_0}$ with full information can be represented as*

$$\{0\} \xrightarrow{a} \{1,2\} \xrightarrow{a,c} \{3,4\} \overset{a}{\circlearrowleft}.$$

Now according to the definition of $\overline{W}_1$ we have

- $\{0\}a\{1,2\}c(\{3,4\}a)^\omega \in \overline{W}_1$ *but*
- $\{0\}a\{1,2\}a(\{3,4\}a)^\omega \notin \overline{W}_1$

and so the winning condition $\overline{W}_1$ is not position based.

The following proposition lists some basic properties of the powerset construction which we will use frequently for proving its correctness. We do not prove the proposition, since the proof is merely technical and does not give much insight into the construction. Nevertheless we mention the proof structure to make clear how the single propositions are related to each other. Proposition (1) is only needed to prove proposition (2) and proposition (2) is the aforementioned property of the edge relation of $\overline{G}_{v_0}$. We need proposition (2) to prove the second part of proposition (4). The first part of propostion (4) is obtained by a successive application of (3), where (3) itself, while very easy to prove, is the key property for the correctness of the powerset construction.

Proposition 3.1. *Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a game with partial information, let $v_0 \in V$ and let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in\overline{A}}, \overline{W}_0)$ be the corresponding game with full information.*

(1) *For* $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ *with* $v(\pi) = v(\pi')$ *we have* $v(\pi a w) = v(\pi' a w)$ *for all* $a \in A$ *and all* $w \in V$ *such that* $\pi a w, \pi' a w \in P_{\mathrm{fin}}(v_0)$.

(2) *If* $\pi \in P_{\mathrm{fin}}(v_0)$ *and* $(v(\pi), \overline{w}) \in \overline{E}_a$ *then we have* $(v(\pi), \overline{w}) = (v(\pi'), v(\pi' b w))$ *for some* $\pi' \in P_{\mathrm{fin}}(v_0)$, *some* $w \in V$ *and some* $b \in A$ *such that* $b \sim_1^A a$ *and* $\pi' \sim_1^* \pi$.

(3) *If* $(\overline{v}, \overline{w}) \in \overline{E}_a$ *then for all* $w \in \overline{w}$ *there exist* $b \in A$ *and* $v \in \overline{v} \cap \mathrm{dom}(f_b)$ *such that* $f_b(v) = w$ *and* $a \sim_1^A b$. *If* $\overline{v} \in \overline{V}_1$, *then* $b = a$.

(4) *For each finite prefix* $\overline{\pi} = \overline{v}_0 a_0 \ldots a_{n-1} \overline{v}_n \in P_{\mathrm{fin}}(\overline{v}_0)$ *of a play in* $\overline{G}_{v_0}$ *from* $\overline{v}_0 = \{v_0\}$ *and all* $v_n \in \overline{v}_n$ *there is a prefix* $\pi = v_0 a'_0 \ldots a'_{n-1} v_n \in P_{\mathrm{fin}}(v_0)$ *of a play in* G *from* v_0 *such that* $a'_i \sim_1^A a_i$ *for* $0 \leq i \leq n-1$ *and* $v_i \in \overline{v}_i$ *for* $0 \leq i \leq n$.
Furthermore, for each such prefix we have $v(v_0 a'_0 \ldots a'_{i-1} v_i) = \overline{v}_i$ *for all* i.

Theorem 3.1. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a game with partial information, let* $v_0 \in V$ *and let* $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{W}_0)$ *be the corresponding game with full information.*

(1) *If there is a terminal position* $\overline{v} \in \overline{V}_1$, *then player* 1 *does not have a strategy for* $\mathcal{G}$ *from initial position* v_0.

(2) *If there is no terminal position* $\overline{v} \in \overline{V}_1$, *then player* 1 *has a winning strategy for* $\mathcal{G}$ *from* v_0 *if and only if he has a winning strategy for* $\overline{G}_{v_0}$ *from* $\overline{v}_0 = \{v_0\}$.

Proof. (1) Let $\overline{v} = v(\pi) = \{\mathrm{last}(\pi') \mid \pi' \in P_{\mathrm{fin}}(v_0),\ \pi' \sim_1^* \pi\ \}$ for some $\pi \in P_{\mathrm{fin}}(v_0)$. Since $\overline{v}$ is a terminal position in the game $\overline{G}_{v_0}$ there is no action $a \in A$ which is available at all positions in $\overline{v}$. But since $\overline{v} \in \overline{V}_1$, that means, $\overline{v} \subseteq V_1$, each strategy for player 1 for $\mathcal{G}$ from v_0 has to be defined and constant on the set $\{\pi' \in P_{\mathrm{fin}}(v_0) \mid \pi' \sim_1^* \pi\}$ which is not possible. So there cannot be any such strategy.

(2) First let $f : \{\pi \in P_{\mathrm{fin}}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ be a winning strategy for player 1 for $\mathcal{G}$ from v_0. We define the strategy $\overline{f} : \{\overline{\pi} \in P_{\mathrm{fin}}(\overline{v}_0) \mid \mathrm{last}(\overline{\pi}) \in \overline{V}_1\} \to A_1 \subseteq \overline{A}$ for player 1 for $\overline{G}_{v_0}$ as follows. For each finite prefix $\overline{\pi} = \overline{v}_0 a_0 \ldots a_{n-1} \overline{v}_n \in P_{\mathrm{fin}}(\overline{v}_0)$ with $\overline{v}_n \in \overline{V}_1$ there is a finite prefix $\pi = v_0 a'_0 \ldots a'_{n-1} v_n \in P_{\mathrm{fin}}(v_0)$ such that $a'_i \sim_1^A a_i$ for $0 \leq i \leq n-1$ and $v_i \in \overline{v}_i$ for $0 \leq i \leq n$. Since $v_n \in \overline{v}_n \in \overline{V}_1$ we have $v_n \in V_1$ and we define $\overline{f}(\overline{\pi}) = f(\pi)$.

Now $v(\pi) = \overline{v}_n$ and since $f(\pi') = f(\pi) =: a$ for all $\pi' \sim_1^* \pi$ we have $a \in \bigcap\{\mathrm{act}(v) \mid v \in v(\pi)\}$. So $(\overline{v}_n, v(\pi a f_a(\mathrm{last}(\pi)))) = (v(\pi), v(\pi a f_a(\mathrm{last}(\pi)))) \in \overline{E}_a^1$. Furthermore the definition is independent of the chosen play prefix $v_0 a'_0 \ldots a'_{n-1} v_n$ since for all $w_0 b_0 \ldots b_{n-1} w_n \in P_{\mathrm{fin}}(v_0)$ with $b_i \sim_1^A a_i$ for $0 \leq i \leq n-1$ and $w_i \in \overline{v}_i$ for $0 \leq i \leq n$ we have $w_0 b_0 \ldots b_{n-1} w_n \sim_1^* \pi$ and thus $f(w_0 b_0 \ldots b_{n-1} w_n) = f(\pi)$.

Now assume that there is a play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ in $\overline{G}_{v_0}$ from initial position $\overline{v}_0$ that is compatible with $\overline{f}$ and that is not won by player 1, that means, $\overline{\pi} \notin \overline{W}_1$. Then according to the definition of $\overline{W}_1$ there is a play $\pi = v_0 a'_0 v_1 a'_1 \ldots \in P(v_0)$ of G from initial position v_0 with $a'_i \sim_i^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$ such that $\pi \notin W_1$. The play π is compatible with f, since if $i < \omega$ such that $v_i \in V_1$, then $\overline{v}_i \in \overline{V}_1$ and so $a'_i = a_i = \overline{f}(\overline{v}_0 a_0 \ldots a_{i-1} \overline{v}_i)$. Furthermore, since $v_0 a'_0 \ldots a'_{i-1} v_i$ is a finite prefix of a play in G from initial position v_0 with $a'_j \sim_1^A a_j$ for all $j < i$ and $v_j \in \overline{v}_j$ for all $j \leq i$ we have $a'_i = a_i = \overline{f}(\overline{v}_0 a_0 \ldots a_{i-1} \overline{v}_i) = f(v_0 a'_0 \ldots a'_{i-1} v_i)$. But this is a contradiction to the fact that f is a winning strategy for player 1 for $\mathcal{G}$ from v_0 and so each play in $\overline{G}_{v_0}$ from $\overline{v}_0$ that is compatible with $\overline{f}$ is won by player 1. Thus $\overline{f}$ is a winning strategy for player 1 for $\overline{G}_{v_0}$ from initial position $\overline{v}_0$.

Now let conversely $\overline{f} : \{\overline{\pi} \in P_{\text{fin}}(\overline{v}_0) \mid \text{last}(\overline{\pi}) \in \overline{V}_1\} \to A_1 \subseteq \overline{A}$ be a winning strategy for player 1 for $\overline{G}_{v_0}$ from initial position $\overline{v}_0$. We define the strategy $f : \{\pi \in P_{\text{fin}}(v_0) \mid \text{last}(\pi) \in V_1\} \to A_1$ for player 1 for $\mathcal{G}$ as follows. For $\pi = v_0a_0v_1a_1\dots a_{n-1}v_n \in P_{\text{fin}}(v_0)$ with $v_n \in V_1$ let $\overline{v}_i := v(v_0a_0v_1a_1\dots a_{i-1}v_i)$ for $i \leq n$. First consider the case that there is some $i < n$ such that $\overline{v}_i \in \overline{V}_1$ and $a_i \notin \mathit{act}(\overline{v}_i)$. Then we choose some $a \in \text{act}(\overline{v}_n)$ and we let $f(\pi') = a$ for all $\pi' \sim_1^* \pi$. If there is no such i, then $\overline{\pi} := \overline{v}_0a_0'\overline{v}_1a_1'\dots a_{n-1}'\overline{v}_n \in P_{\text{fin}}(\overline{v}_0)$ holds, where $a_i' \in \overline{A}$, $i = 0, \dots, n-1$ are the uniquely determined actions with $a_i' \sim_1^A a_i$ and we define $f(\pi) := \overline{f}(\overline{\pi})$.

First, f is a strategy for player 1 for $\mathcal{G}$ since if $\pi = v_0a_0\dots a_{n-1}v_n$, $\pi' = w_0b_0\dots b_{n-1}w_n \in P_{\text{fin}}(v_0)$ with $\pi \sim_1^* \pi'$ and $v_n, w_n \in V_1$ then $\overline{v}_i = \overline{w}_i$ for $0 \leq i \leq n$ and $a_i \sim_1^A b_i$ for all $0 \leq i \leq n-1$. So either there is some $i < n$ such that $\overline{v}_i = \overline{w}_i \in \overline{V}_1$ and $a_i = b_i \notin \mathit{act}(\overline{v}_i) = \mathit{act}(\overline{w}_i)$. Then by definition of f we have $f(\pi) = f(\pi')$. Or we have $f(\pi) = \overline{f}(\overline{v}_0a_0'\dots a_{n-1}'\overline{v}_n)$ and $f(\pi') = \overline{f}(\overline{w}_0b_0'\dots b_{n-1}'\overline{w}_n)$ where $a_i' = b_i'$ for $0 \leq i \leq n-1$ and due to $\overline{v}_i = \overline{w}_i$ for $0 \leq i \leq n$ we have $f(\pi) = f(\pi')$.

Now assume that there is a play $\pi = v_0a_0v_1a_1\dots \in P(v_0)$ in G from initial position v_0 that is compatible with f and not won by player 1 and let $a_i' \in \overline{A}$ for $i < \omega$ be the uniquely determined actions with $a_i' \sim_1^A a_i$.

Now if $i < \omega$ with $\overline{v}_i \in \overline{V}_1$, then $v_i \in \overline{v}_i \subseteq V_1$ and $a_i \in \text{act}(v_i) \subseteq A_1$. Since $a_i \sim_1^A a_i'$ and $a_i' \in \overline{A}$ we have $a_i' = a_i$. Furthermore, π is compatible with f and so the definition of f yields $a_i' = a_i = f(v_0a_0\dots a_{i-1}v_i) = \overline{f}(\overline{v}_0a_0'\dots a_{i-1}'\overline{v}_i) \in \text{act}(\overline{v}_i)$, because the actions $a_0', \dots, a_i'$ are uniquely determined by $a_i' \sim_1^A a_i$. So $\overline{\pi} = \overline{v}_0a_0'\overline{v}_1a_1'\dots$ is a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ which is compatible with $\overline{f}$ and thus, $\overline{\pi}$ is won by player 1. But since $\pi \notin W_1$ with $a_i \sim_1^A a_i'$ and $v_i \in v_i^*$ for all $i < \omega$ we have $\overline{\pi} \notin \overline{W}_1$ which is a contradiction. Thus, each play in $\mathcal{G}$ from v_0 that is compatible with f is won by player 1. ☐

Corollary 3.1. *Let* $\mathcal{G} = (G, (\text{vis}_i^V), (\text{vis}_i^A))$, $G = (V, V_0, (f_a)_{a\in A}, W_0)$ *be a game with partial information. Then there is a winning condition* $W_0' \supseteq W_0$ *such that* $W_1' = P(V) \setminus W_0'$ *is a partial information winning condition with respect to* $\sim_1^\omega$ *and such that for each* $v_0 \in V$, *player 1 has a winning strategy for* $\mathcal{G}$ *from* v_0 *if and only if he has a winning strategy for* $\mathcal{G}' = (G', (\text{vis}_i^V), (\text{vis}_i^A))$ *from* v_0, *where* $G' = (V, V_0, (f_a)_{a\in A}, W_0')$.

Proof. Define $W_0' := \{\pi \in P(V) \mid \exists \pi' \in P(V) : \pi \sim_1^\omega \pi' \text{ and } \pi' \in W_0\}$. Now let $v_0 \in V$ and consider the corresponding games $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{W}_0)$ and $\overline{G}'_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{W}'_0)$ with full information. Using the definition of $\overline{W}_0$ one can easily show that for each $\overline{\pi} \in P(\overline{v}_0)$ where $\overline{v}_0 = \{v_0\}$, we have $\overline{\pi} \in \overline{W}_0$ if and only if $\overline{\pi} \in \overline{W}'_0$. Now by Theorem 3.1, the proof is finished. ☐

Remark. In general, for a game $\mathcal{G} = (G, (\text{vis}_i^V)), (\text{vis}_i^A))$, $G = (V, V_0, (f_a)_{a\in A}, W_0)$ with partial information and a position $v_0 \in V$, there is no deterministic game

$H = (V', V_0', (g_a)_{a \in A'}, W_0')$ and a position $v_0' \in V'$ such that *for all* $i \in \{0,1\}$, player i has a winning strategy for $\mathcal{G}$ from v_0 if and only if he has a winning strategy for H from v_0'. The reason is that deterministic games with full information are determined while games with partial information are not determined in general.

3.1.1 Observation Based Winning Conditions

In this section we consider the special case of observation based winning conditions. If the game graph of the game G is finitely branching, then the winning condition $\overline{W}_1$ is position based and we can describe the winning condition explicitly by means of the sequences of observations which belong to W_1. First we have to say what we mean by this.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information and observation based winning condition W_1. Furthermore let $v_0 \in V$ and let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{W}_0)$ be the corresponding game with full information. For $\pi = v_0 a_0 v_1 a_1 \ldots \in V(AV)^* \cup V(AV)^\omega$ we call $\mathrm{obs}_1(\pi) = \mathrm{vis}_1^V(v_0)\,\mathrm{vis}_1^V(v_1)\ldots$ the sequence of observations of player 1 in π. Now since W_1 is observation based we have $\pi \in W_1$ if and only if $\mathrm{obs}_1(\pi) \in \mathrm{obs}_1(W_1) := \{\mathrm{obs}_1(\pi) \mid \pi \in W_1\}$ for each $\pi \in P(V)$. Furthermore, for a set $S \subseteq V$ with $u \sim_1^V v$ for all $u, v \in S$ we define $\mathrm{vis}_1^V(S) := \mathrm{vis}_1^V(v)$ for some $v \in S$. Finally, for a sequence $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in \overline{V}(\overline{A}\,\overline{V})^\omega$ we define $\mathrm{obs}_1(\overline{\pi}) := \mathrm{vis}_1^V(\overline{v}_0)\,\mathrm{vis}_1^V(\overline{v}_1)\ldots$

Proposition 3.2. *If for all* $v \in V$ *the set* $\{f_a(v) \mid a \in \mathrm{act}(v)\}$ *is finite, then for each play* $\overline{\pi} \in P(\overline{v}_0)$ *of* $\overline{G}_{v_0}$ *we have* $\overline{\pi} \in \overline{W}_1$ *if and only if* $\mathrm{obs}_1(\overline{\pi}) \in \mathrm{obs}_1(W_1)$.

Proof. Let $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ and let first $\mathrm{obs}_1(\overline{\pi}) \in \mathrm{obs}_1(W_1)$. Then for each play $\pi = v_1 a_1' v_2 a_2' \ldots \in P(v_0)$ of $\mathcal{G}$ from v_0 with $v_i \in \overline{v}_i$ for all $1 \leq i < \omega$ we have $\mathrm{obs}_1(\pi) = \mathrm{obs}_1(\overline{\pi}) \in \mathrm{obs}_1(W_1)$ and thus $\pi \in W_1$. So by definition of $\overline{W}_1$ we have $\overline{\pi} \in \overline{W}_1$.

Now let conversely $\overline{\pi} \in \overline{W}_1$. We define $V^t = \bigcup\{\overline{v}_i \times \{i\} \mid i < \omega\}$ and for $a \in A$ we define the edge relation $(E_a^t)' \subseteq V^t \times V^t$ by $((u,i),(v,i+1)) \in (E_a^t)'$ if and only if $a \in \mathrm{act}(u)$, $f_a(u) = v$ and $a \sim_1^A a_i$. Furthermore let $(E^t)' := \bigcup\{(E_a^t)' \mid a \in A\}$ and let E^t be obtained from $(E^t)'$ by deleting for each $(v,i) \in V^t$ which has more than one predecessor with respect to $(E^t)'$ all the edges but one from predecessors of (v,i) to (v,i). (It does not matter which edge we keep.)

Clearly this yields a tree t with root $(v_0, 0)$ and successor relation given by E^t. Now using Proposition 3.1, by a simple induction over i we see that for all $i < \omega$ and all $u \in \overline{v}_i$, the node (u,i) occurs in t (at level i). So t is a finite branching infinite tree and thus, due to König's Lemma there is an infinite path $(v_0,0)(v_1,1)\ldots$ in t. This path yields a play $\pi = v_0 a_0' v_1 a_1' \ldots \in P(v_0)$ of G from initial position v_0 with $a_i' \sim_1^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$. Since $\overline{\pi} \in \overline{W}_1$, by definition of $\overline{W}_1$ we have $\pi \in W_1$ and thus $\mathrm{obs}_1(\overline{\pi}) = \mathrm{obs}_1(\pi) \in \mathrm{obs}_1(W_1)$. ☐

Information compatible Muller-conditions. Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_1^A))$ with $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0))$ be a partial information Muller-game. We call $\mathcal{G}$ an information compatible Muller-game, if the coloring col is compatible with the information of player 1. (Notice that we are only asking for winning strategies of player1.) Then clearly the Muller-condition W_1 is observation based and so if we assume that for each $v \in V$ the set $\{f_a(v) \mid a \in \mathrm{act}(v)\}$ is finite, we can apply Proposition 3.2 to $\mathcal{G}$. If we do so, we can easily see that the winning condition of $\overline{G}_{v_0}$ is again a Muller-condition with the following coloring and winning component $\mathcal{F}_0$.

$$\overline{\mathrm{col}} : \overline{V} \to C \quad \text{with} \quad \overline{\mathrm{col}}(\overline{v}) = \mathrm{col}(v) \text{ for some } v \in \overline{v}.$$

Of course, this construction also transforms parity conditions into parity conditions and (co-) Büchi-conditions into (co-) Büchi-conditions.

Furthermore, in the same way, partial information games with information compatible (co-) reachability conditions are transformed into games with (co-) reachability conditions. Nevertheless, as we will see in the next section, Proposition 3.2 does not hold in general, if the game graph is not finitely branching, even for safety conditions.

3.1.2 Reachability and Safety Games

First we consider an information compatible safety game with infinitely branching game graph for which Proposition 3.2 does not hold.

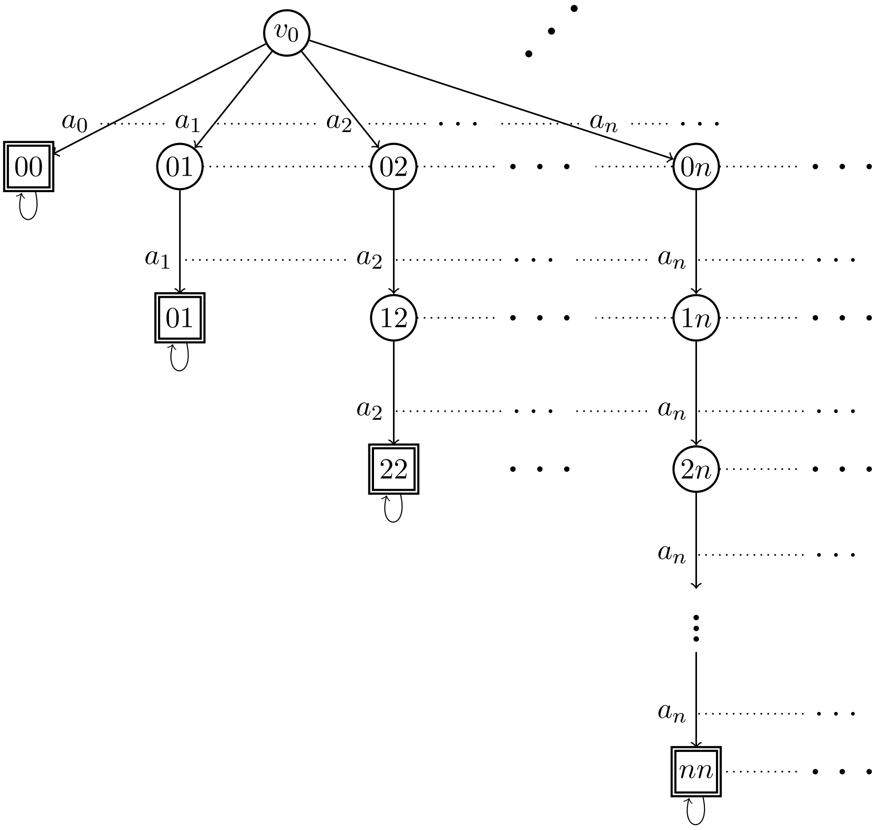

Figure 3.2: An infinite branching safety game.

Example 3.2. *We define the safety game $G = (V, V_0, (f_a)_{a\in A}, R)$ as follows. $V = \{v_0\} \cup \{(i,j) \in \omega \times \omega \,|\, i \leq j\}$, $V_0 = R = V \setminus \{(i,i) \,|\, i < \omega\}$ and $A = \{a_i \,|\, i < \omega\} \cup \{\circlearrowleft\}$. The availability of actions is given as follows.*

- *$v_0 \xrightarrow{a_j} (0,j)$ for all $j < \omega$ and $(i,i) \xrightarrow{\circlearrowleft} (i,i)$ for all $i < \omega$.*
- *$(i,j) \xrightarrow{a_j} (i+1,j)$ for all $j \in \omega$ and all $i < j$.*

The information of player 1 is defined by $\mathrm{vis}_1^V(v_0) = v_0$, $\mathrm{vis}_1^V(i,j) = i$ for all $j < \omega$ and all $i < j$ and $\mathrm{vis}_1^V(i,i) = (i,i)$ for all $i < \omega$. Furthermore, $\mathrm{vis}_1^A(a_i) = a$ for all $i < \omega$ and $\mathrm{vis}_1^A(\circlearrowleft) = \circlearrowleft$. The game is delineated in Figure 3.2.

The safety condition which is given by R corresponds to the coloring $\mathrm{col} : V \to \{0,1\}$ where we have $\mathrm{col}(i,j) = 0$, if $i < j$ and $\mathrm{col}(i,i) = 1$ for all $i < \omega$ and the winning component $\mathcal{F}_0 = \{\{0\}\}$. Obviously we have $W_1 = P(v_0)$.

Now consider the corresponding game $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in\overline{A}}, \overline{W}_0)$ with full information. There we have $\overline{V} = \{\overline{v}_0 := \{v_0\}, \overline{w}_0 := \{(0,0)\}\} \cup \{\overline{v}_i, \overline{w}_i \,|\, 1 \leq i < \omega\}$ with $\overline{v}_i = \{(i-1,j) \,|\, i \leq j < \omega\}$ and $\overline{w}_i = \{(i,i)\}$ for all $1 \leq i < \omega$ and $\overline{V}_0 = \{\overline{w}_i \,|\, 1 \leq i < \omega\}$. Furthermore we can choose $\overline{A} = \{a, \circlearrowleft\}$ and for each $i < \omega$ we have $\overline{v}_i \xrightarrow{a} \overline{w}_i$ and $\overline{v}_i \xrightarrow{a} \overline{v}_{i+1}$. Finally we have $\overline{w}_i \xrightarrow{\circlearrowleft} \overline{w}_i$ for all $i < \omega$.

Obviously $P(\overline{v}_0) \subseteq \overline{W}_1$. But now for the safety set $\overline{R}$ we have $\overline{R} = \{\overline{v}_i \,|\, i < \omega\}$. So the play $\overline{\pi} = \overline{v}_0 a \overline{v}_1 a \overline{v}_2 \ldots$ is won by player 0 according to the safety condition which is given by $\overline{R}$. So it does not coincide with $\overline{W}_0$.

Remark 1. In fact, player 1 has a winning strategy for the game $\mathcal{G}$ from initial position v_0 while he does not have a winning strategy for the game $\overline{G}_{v_0}$ from initial position $\overline{v}_0$, if we consider the game as a safety game with safety set $\overline{R}$. So this condition is not the right winning condition for the game $\overline{G}_{v_0}$.

Remark 2. By a somewhat extended version of this example one can show that there is a game with observation based winning condition such that the winning condition of the corresponding game with full information is not position based.

Now consider an *arbitrary* partial information (co-) reachability game

$$\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A)) \text{ with } G = (V, V_0, (f_a)_{a\in A}, R).$$

First, if for $i \in \{0,1\}$ we have $\mathrm{act}(u) = \mathrm{act}(v)$ for all $u, v \in V_i$ with $u \sim_i^V v$, we can transform $\mathcal{G}$ into a information compatible (co-) reachability game such that the existence of winning strategies for both players is preserved from each position as follows. For $i \in \{0,1\}$ let $(vis')_i^V(v) := \mathrm{vis}_i^V(v)$ for all $v \in V \setminus R$ and $(vis')_i^V(v) = v$, for all $v \in R$, where we assume that $v \notin \mathrm{VIS}_i^V$. Now let $\mathcal{G}' = (G, ((vis')_i^V), (\mathrm{vis}_i^A))$.

Clearly, if $f : \{\pi \in P_{\mathrm{fin}}(v_0) \,|\, \mathrm{last}(\pi) \in V_i\} \to A$ is a winning strategy for $\mathcal{G}$ from some initial position $v_0 \in V$, then f is a winning strategy for player i for $\mathcal{G}'$ from v_0 as well.

If conversely player i has a winning strategy $g : \{\pi \in P_{\mathrm{fin}}(v_0) \,|\, \mathrm{last}(\pi) \in V_i\} \to A$ for $\mathcal{G}'$ from some initial position $v_0 \in V$, then we define the strategy $f : \{\pi \in P_{\mathrm{fin}}(v_0) \,|\, \mathrm{last}(\pi) \in V_i\} \to A$ for player i for $\mathcal{G}$ from v_0 as follows. For $\pi = v_0 a_0 \ldots a_{n-1} v_n \in P_{\mathrm{fin}}(v_0)$ with $\mathrm{last}(\pi) \in V_i$ such that $\mathrm{occ}(\pi) = \{v_0, \ldots, v_n\} \cap R = \emptyset$

we define $g(\pi) := f(\pi)$. For $\pi \in P_{\text{fin}}(v_0)$ with $\text{last}(\pi) \in V_i$ such that $\text{occ}(\pi) \cap R \neq \emptyset$ we let $g(\pi)$ be an arbitrary action from $\text{act}(\text{last}(\pi))$ while regarding the condition that g has to be constant over equivalence classes of finite play prefixes. (Clearly this is possible since $\text{act}(u) = \text{act}(v)$ for all $u, v \in V_1$ with $u \sim_1^V v$.) Then obviously, g is a winning strategy for player i for $\mathcal{G}$ from v_0.

Now we consider the reachability case. Let $v_0 \in V$ and consider the corresponding game $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{W}_0)$ with full information. We show that Theorem 3.1 holds for the reachability game $H = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, \overline{R})$ as well, where $\overline{R}$ is defined by $\overline{v} \in \overline{R}$ if $\overline{v} \cap R \neq \emptyset$. Notice that the reachability condition which is defined by $\overline{R}$ does not necessarily coincide with the winning condition $\overline{W}_0$, even if $\mathcal{G}$ is information compatible. (Consider for instance the game from Example 3.2 as a reachability game with $R = V$.)

With the same arguments as in the proof of Theorem 3.1, if there is a terminal position $\overline{v} \in \overline{V}_1$, then player 1 does not have any strategy for $\mathcal{G}$ from v_0. Now we show that if there is no such terminal position, then player 1 has a winning strategy for $\mathcal{G}$ from v_0 if and only if he has a winning strategy for H from $\overline{v}_0 = \{v_0\}$.

So let first $\overline{f} : \{\overline{\pi} \in P_{\text{fin}}(\overline{v}_0) \mid \text{last}(\overline{\pi}) \in \overline{V}_1\} \to \overline{A}$ be a winning strategy for player 1 for H from $\overline{v}_0$. We define $f : \{\pi \in P_{\text{fin}}(v_0) \mid \text{last}(\pi) \in V_1\} \to A$ just as in the proof of Theorem 3.1. Now assume that there is a play $\pi = v_0 a_0 v_1 a_1 \ldots \in P(v_0)$ that is compatible with f and not won by player 1, that means, there is an index $i < \omega$ such that $v_i \in R$. Then according to the definition of $\overline{R}$ we have $\overline{v}_i \in \overline{R}$, where the play $\overline{\pi} = \overline{v}_0 a'_0 \overline{v}_1 \ldots \in P(\overline{v}_0)$ is defined as in the proof of Theorem 3.1. So $\overline{\pi}$ is won by player 0 and $\overline{\pi}$ is compatible with $\overline{f}$ which is a contradiction.

Now let $f : \{\pi \in P_{\text{fin}}(v_0) \mid \text{last}(\pi) \in V_1\} \to A$ be a winning strategy for player 1 for $\mathcal{G}$ from v_0. We define $\overline{f} : \{\overline{\pi} \in P_{\text{fin}}(\overline{v}_0) \mid \text{last}(\overline{\pi}) \in \overline{V}_1\} \to \overline{A}$ just as in the proof of Theorem 3.1. Now assume that there is a play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ which is compatible with $\overline{f}$ and not won by player 1.

Then there is an index $i < \omega$ such that $\overline{v}_i \in \overline{R}$, that means, $v_i \in \overline{v}_i$ for some $v_i \in R$. According to Proposition 3.1 there is a finite play prefix $\pi_{\text{fin}} = v_0 a'_0 \ldots a'_{i-1} v_i \in P_{\text{fin}}(v_0)$ with $v_j \in v_j^*$ for $j = 0, \ldots, i$ such that π_{fin} is compatible with f. But then obviously we can extend π_{fin} to a play $\pi = \pi_{\text{fin}} a'_i v_{i+1} a'_{i+1} \ldots \in P(v_0)$ which is compatible with f as well and which is not won by player 1. This is a contradiction and therefore, $\overline{f}$ is a winning strategy for player 1 for H from $\overline{v}_0$.

3.1.3 Omega-Regular Winning Conditions

In this section we study games with partial information where the winning condition is omega-regular, that means, given by an S1S-formula or a nondeterministic Büchi-automaton. We show that in this case, the winning condition of the corresponding game with full information is again omega-regular and we can effectively construct an automaton recognizing this winning condition, where the state set is roughly $Q \times V$ (with Q the state set of the original automaton and V the set of positions of the game). We also construct an S1S-formula defining the winning condition of the game with full information from the formula defining the winning condition of the original game, without using the translation of formulas into automata and back.

The point here is that we can solve games with full information, where the winning condition is given by a deterministic parity automaton by taking the product of the game graph and the automaton and solving the resulting parity game. However, if the winning condition of a game with partial information is given by a deterministic parity automaton, then the winning condition of the corresponding game with full information is given by a parity automaton which is not deterministic in general. In order to solve the game we have to construct an equivalent deterministic parity automaton which may cause a high complexity.

Omega-Automata. An ω-automaton has the form $\mathcal{A} = (\Sigma, Q, q_0, \Delta, \mathrm{acc})$, where Σ is a finite alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation and $\mathrm{acc} \subseteq Q^\omega$ is the acceptance component. A run of $\mathcal{A}$ on an ω-word $\alpha \in \Sigma^\omega$ is a function $\rho \in Q^\omega$ such that $\rho(0) = q_0$ and $(\rho(i), \alpha(i), \rho(i+1)) \in \Delta$ for all $i < \omega$. The run is accepting, if $\rho \in \mathrm{acc}$. The automaton $\mathcal{A}$ accepts an ω-word $\alpha \in \Sigma^\omega$ if there is an accepting run of $\mathcal{A}$ on α. We define $L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}$. The automaton is called deterministic, if for all $(q, a) \in Q \times \Sigma$ there is exactly one transition $(q, a, q') \in \Delta$, that means, Δ is a function $Q \times \Sigma \to Q$.

In the same way as we have defined Muller-games, we define Muller-automata (and the corresponding special cases) by a coloring $\mathrm{col} : Q \to C$ for some finite set $C \subseteq \omega$ and a set $\mathcal{F} \subseteq 2^C$ (cf. Section 2.6.1). An ω-language $L \subseteq \Sigma^\omega$ is called regular, if there is a Büchi-automaton $\mathcal{A}$ over the alphabet Σ such that $L = L(\mathcal{A})$.

A full proof of the following theorem can be found in [GTW02]. The hard part of the proof is to show that all regular ω-languages are deterministicly Muller-recognizable. This is included in the Muller-Schupp construction for removing alternation from tree automata, see [MS95]. In [GTW02] an earlier construction by Safra is used.

Theorem 3.2. *For an ω-language $L \subseteq \Sigma^\omega$, the following statements are equivalent.*

(1) L is ω-regular.
(2) L is deterministicly Muller-recognizable.
(3) L is deterministicly parity recognizable.

A finite game $G = (V, V_0, (f_a)_{a \in A}, W_0)$ is called ω-regular, if the set $W_0 \subseteq (VA)^\omega$ is regular. Notice that of course finite Muller-games are ω-regular since we can take the set of colors of the game as state set of the automaton to obtain a deterministic Muller-automaton recognizing the winning condition. Then we can translate this automaton into a nondeterministic Büchi-automaton.

S1S. S1S is the monadic second order logic over words (S1S stands for second order logic of one successor). For an alphabet Σ we define the signature $\tau_\Sigma = \{S, \min, <, (P_a)_{a \in \Sigma}\}$ and for an ω-word $\alpha \in \Sigma^\omega$ we define the τ_Σ- structure $\underline{\alpha} = (\omega, S, \min, <, (P_a^\alpha)_{a \in \Sigma})$ where S is the successor function on ω, min and $<$ are as usual and $P_a^\alpha = \{i < \omega \mid \alpha(i) = a\}$ for all $a \in \Sigma$. Now for an $\mathrm{MSO}(\tau_\Sigma)$-sentence φ we define $L(\varphi) = \{\alpha \in \Sigma^\omega \mid \underline{\alpha} \models \varphi\}$ and we say that a language $L \subseteq \Sigma^\omega$ is S1S-definable, if there is an $\mathrm{MSO}(\tau_\Sigma)$-sentence φ such that $L = L(\varphi)$. For a formal definition of monadic second order logic MSO, see Section 4.3.3.

Theorem 3.3. *(Büchi, 1962) An omega-language L is regular if and only if it is S1S-definable.*

A proof of this theorem can be found for example in [Tho90]. The translations from automata into logical formulas and vice versa are effective, but the inductive translation of logical formulas into automata has a nonelementary complexity. (The reason is that each negation symbol in the formula corresponds to complementing the automaton and thus may cause an exponential blow up of the state space.) So if the winning condition of a game with partial information is given by a logical formula and we are only interested in a logical formula defining the winning condition of the game with full information, then of course we do not want to use automata as an intermediate step.

Now let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a finite game with partial information and let $\mathcal{A} = (VA, Q, q_0, \Delta, \mathrm{acc})$ be an ω-automaton recognizing W_0. Furthermore let $v_0 \in V$ and let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{W}_0)$ be the corresponding game with full information. Then for each play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ in $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$ we have $\overline{\pi} \in \overline{W}_0$ if and only if there is some play $\pi = v_0 a'_0 v_1 a'_1 \in P(v_0)$ in G from v_0 with $a'_i \sim_1^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$ such that $\pi \in W_0$. Now the automaton that accepts $\overline{W}_0$, guesses such a play π and at the same time simulates $\mathcal{A}$ on π.

Formally, we define $\mathcal{B} = (\overline{V}A, \overline{Q}, (q_0, v_0), \overline{\Delta}, \overline{\mathrm{acc}})$ as follows.

- $\overline{Q} = Q \times V$.
- A sequence from $\overline{Q}^\omega$ is in $\overline{\mathrm{acc}}$ if and only if the corresponding sequence from Q^ω (where we eliminate all second components) is in acc.
- $((p, v), \overline{v}a, (q, w)) \in \overline{\Delta} :\Longleftrightarrow$
  $v \in \overline{v}$ and there is some action $b \sim_1^A a$ such that $b \in \mathrm{act}(v)$, $f_b(v) = w$ and $(p, vb, q) \in \Delta$.

Clearly this construction turns Muller-automata into Muller-automata, parity automata into parity automata and Büchi-automata into Büchi-automata.

Proposition 3.3. *For each play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ in $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$ we have $\overline{\pi} \in \overline{W}_0$ if and only if $\overline{\pi} \in L(\mathcal{B})$.*

Proof. First let $\overline{\pi} \in \overline{W}_0$, that means, there is some play $\pi = v_0 a'_0 v_1 a'_1 \ldots \in P(v_0)$ in G from v_0 with $a'_i \sim_1^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$ such that $\pi \in W_0$. Since $W_0 = L(\mathcal{A})$, there is an accepting run $\rho : \omega \to Q$ of $\mathcal{A}$ on π. Now consider $\overline{\rho} : \omega \to \overline{Q}$ with $\overline{\rho}(i) := (\rho(i), v_i)$ for $i < \omega$. By definition of $\mathcal{B}$, $\overline{\rho}$ is a run of $\mathcal{B}$ on $\overline{\pi}$ and since ρ is accepting, so is $\overline{\rho}$.

Now let conversely $\overline{\pi} \in L(\mathcal{B})$ and let $\overline{\rho} : \omega \to \overline{Q}$ be an accepting run of $\mathcal{B}$ on $\overline{\pi}$. We define $\rho : \omega \to Q$ by $\rho(i) := \mathrm{pr}_1(\overline{\rho}(i))$ for $0 < i < \omega$. Since $\overline{\rho} \in \overline{\mathrm{acc}}$ we have $\rho \in \mathrm{acc}$. Furthermore, we define a sequence $\pi^0 \preceq \pi^1 \preceq \pi^2 \preceq \ldots$ of prefixes $\pi^i = v_0 a'_0 v_1 a'_1 \ldots a'_{i-1} v_i \in P_{\mathrm{fin}}(v_0)$ with $v_i = \mathrm{pr}_2(\overline{\rho}(i))$ and $a'_{i-1} \sim_1^A a_{i-1}$ for $i < \omega$, such that $\rho(\leq i)$ is a finite prefix of a run of $\mathcal{A}$ on some extension of π^i to a play in G for all $i < \omega$.

First we define $\pi^0 := v_0$. Now let $0 < i < \omega$. Then $(\overline{\rho}(i), \overline{v}_i a_i, \overline{\rho}(i+1)) \in \overline{\Delta}$, that means, $v \in \overline{v}_i$ and there exists $b \sim_1^A a_i$ such that $b \in \operatorname{act}(v)$, $f_b(v) = w$ and $(p, vb, q) \in \Delta$, where $\overline{\rho}(i) = (p, v)$ and $\overline{\rho}(i+1) = (q, w)$. So we define $\pi^i := \pi^{i-1} bw$. Notice that $\rho(i) = q$, $\operatorname{last}(\pi^{i-1}) = v$ and $\rho(i+1) = q$.

Now the sequence $\pi \in V(AV)^\omega$ with $\operatorname{first}(\pi) = v_0$ and $\pi(i) = \pi^i(i)$ for all $0 < i < \omega$ is a play in G from v_0 and ρ is an accepting run of $\mathcal{A}$ on π, so $\pi \in W_0$. Furthermore, $a_i' \sim_1^A a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$. By definition of $\overline{W}_1$ this yields $\overline{\pi} \in \overline{W}_0$. ☐

Solving ω-regular games. Now we want to see how we can solve ω-regular games with full information. For this purpose we present a general result about ω-regular games with partial information. For the case of games with full information, this result yields a method to solve such games.

Let $\mathcal{G} = (G, (\operatorname{vis}_i^V), (\operatorname{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information and let $\mathcal{A} = (VA, Q, q_0, \delta, \operatorname{acc})$ be a deterministic ω-automaton such that $L(\mathcal{A}) = W_0$. Now we define the game $\mathcal{G}' = (G', ((\operatorname{vis}')_i^V), (\operatorname{vis}_i^A)$ with $G' = (V \times Q, V_0 \times Q, (f_a')_{a \in A}, W_0')$ as follows.

- $\operatorname{act}(v, q) = \operatorname{act}(v)$.
- $f_a'((v, q)) := (f_a(v), \delta(q, va))$.
- $\pi = v_0 a_0 v_1 \ldots \in W_0'$ if and only if $\operatorname{pr}_2(v_0) \operatorname{pr}_2(v_1) \ldots \in \operatorname{acc}$.
- $(\operatorname{vis}')_i^V((v, q)) = \operatorname{vis}_i^V(v)$ for $i \in \{0, 1\}$.

Proposition 3.4. *For each $i \in \{0, 1\}$, for all $v \in V$, player i has a winning strategy for $\mathcal{G}$ from v if and only if he has a winning strategy for $\mathcal{G}'$ from (v, q_0).*

Now if $\mathcal{A}$ is a deterministic Muller-automaton with acc defined by a pair $(\operatorname{col}, \mathcal{F})$, then the winning condition W_0' of G' is again a Muller-condition $(\operatorname{col}', \mathcal{F})$ with $\operatorname{col}'((v, q)) := \operatorname{col}(q)$, that means, G' is a Muller-game. In particular, if $\mathcal{A}$ is a parity automaton, then G' is a parity game since the component $\mathcal{F}$ is not changed.

Now let the game G with *full information* be ω-regular. We solve the strategy problem for G as follows. First, we construct a deterministic parity automaton $\mathcal{B}$ with $L(\mathcal{B}) = W_0$. (Such an automaton might already be included in the description of G, but the winning condition might also be given for example by a nondeterministic Büchi-automaton or an S1S-formula.) Then we carry out the above construction and we apply an algorithm for solving the strategy problem for parity games to G'.

Now let W_0 be omega-regular and let φ be an S1S-formula defining W_0 over the signature τ_{VA}, where $VA = \{v^0 a^0, \ldots, v^0 a^m, \ldots, v^n a^0, \ldots, v^n a^m\}$ for some $n, m < \omega$. We define the S1S-formula $\overline{\varphi}$ over the signature $\tau_{\overline{V}A}$ as follows. All pairs $(p, q), (k, l)$ range over $N \times M$ with $N := \{0, \ldots, n\}$ and $M := \{0, \ldots, m\}$.

$$\overline{\varphi} := \quad \exists X_{00} \ldots \exists X_{0m} \ldots \exists X_{n0} \ldots \exists X_{nm} \quad [$$

$$(1)\ \forall x(\bigvee_{(p,q)} (\ X_{pq} x \ \wedge \bigwedge_{(k,l) \neq (p,q)} \neg X_{kl} x \ \wedge \bigvee_{\overline{v}a \in \overline{V}A,\ v^p \in \overline{v},\ a^q \sim_1^A a} P_{\overline{v}a} x \ ) \ ) \ \wedge$$

$$(2)\ \forall x \forall y (S(x) = y \ \rightarrow \bigvee_{(p,q),(k,l)\,:\, a^q \in act(v^p),\ f_{a^q}(v^p) = v^k} X_{pq}x \wedge X_{kl}y) \ \ \wedge$$

$$(3)\ \varphi(P_{v^0a^0}/X_{00}, \ldots, P_{v^0a^m}/X_{0m}, \ldots, P_{v^na^0}/X_{n0}, \ldots, P_{v^na^m}/X_{nm}) \quad]$$

The formula in (3) is obtained from φ by replacing (simultaneously) each occurrence of $P_{v^pa^q}$ with X_{pq} for all $(p, q) \in N \times M$.

Proposition 3.5. *For each play $\overline{\pi} = \overline{v}_0 a_0 \overline{v}_1 a_1 \ldots \in P(\overline{v}_0)$ in $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$ we have $\overline{\pi} \in \overline{W}_0$ if and only if $\underline{\overline{\pi}} \models \overline{\varphi}$.*

Proof. Let $\overline{\psi}(X_{00}, \ldots, X_{nm})$ be the conjunction of the formulas in (1) and (2) and let first $\overline{\pi} \in \overline{W}_0$, that means, there is some play $\pi = v_0 a'_0 v_1 a'_1 \ldots \in P(v_0)$ in G from v_0 with $a'_i \sim^A_1 a_i$ and $v_i \in \overline{v}_i$ for all $i < \omega$ such that $\pi \in W_0$, that means, $\underline{\pi} \models \varphi$.

Now for $(p, q) \in N \times M$ let $P_{pq} := P_{v^pa^q} = \{i < \omega \,|\, v_i a'_i = v^p a^q\}$. Then for any $i < \omega$ we have $i \in P_{v_i a'_i} = P_{pq}$ for $(p, q) \in N \times M$ with $v_i a'_i = v^p a^q$ and for all $(k, l) \in N \times M$ with $(k, l) \neq (p, q)$ we have $i \notin P_{kl}$. Furthermore, $i \in P_{\overline{v}_i a_i}$ and we have $v_i \in \overline{v}_i$ and $a_i \sim^A_1 a'_i$. Finally, $a'_i \in \mathrm{act}(v_i)$, $v_{i+1} = f_{a'_i}(v_i)$ and $i + 1 \in P_{v_{i+1} a'_{i+1}}$. Thus, $\underline{\overline{\pi}} \models \psi(P_{00}, \ldots, P_{nm})$ and $\underline{\pi} \models \varphi(P_{v^0a^0}/P_{00}, \ldots, P_{v^na^m}/P_{nm})$ and therefore $\underline{\overline{\pi}} \models \varphi(P_{v^0a^0}/P_{00}, \ldots, P_{v^na^m}/P_{nm})$. So we can conclude that $\underline{\overline{\pi}} \models \overline{\varphi}$.

Now let conversely $\underline{\overline{\pi}} \models \overline{\varphi}$ and let $P_{pq} \subseteq \omega$ for $(p, q) \in N \times M$ be such that $\underline{\overline{\pi}} \models \overline{\psi}(P_{00}, \ldots, P_{nm})$ and $\underline{\overline{\pi}} \models \varphi(P_{v^0a^0}/P_{00}, \ldots, P_{v^na^m}/P_{nm})$.

We define the sequence $\pi = v_0 a'_0 v_1 a'_1 \ldots \in V(AV)^\omega$ by $(v_i, a'_i) = (v^p, a^q)$ for the uniquely determined $(p, q) \in N \times M$ with $i \in P_{pq}$. (Notice that according to the subformula (1) of $\overline{\psi}$, the sets P_{pq} form a partition of ω.) According to subformula (2) of $\overline{\psi}$, if $i < \omega$, then $i \in P_{pq}$ and $i + 1 \in P_{kl}$ such that $a^q \in \mathrm{act}(v^p)$ and $f_{a^q}(v^p) = v^k$. By definition of π we have $(v^p, a^q) = (v_i, a'_i)$ and $(v^k, a^l) = (v_{i+1}, a'_{i+1})$. So $a'_i \in \mathrm{act}(v_i)$ and $f_{a'_i}(v_i) = v_{i+1}$ for all $i < \omega$ and thus $\pi \in P(v_0)$. Finally, subformula (1) of $\overline{\psi}$ yields that for any $i < \omega$ we have $i \in P_{\overline{v}a}$ such that $v^p \in \overline{v}$ and $a \sim^A_1 a^q$ where $i \in P_{pq}$. Since $(v_i, a'_i) = (v^p, a^q)$ and $\overline{v}_i a_i = \overline{v}a$ this means $v_i \in \overline{v}_i$ and $a'_i \sim^A_1 a_i$. Furthermore, $\underline{\overline{\pi}} \models \varphi(P_{v^0a^0}/P_{00}, \ldots, P_{v^na^m}/P_{nm})$ and thus $\underline{\pi} \models \varphi(P_{v^0a^0}/P_{00}, \ldots, P_{v^na^m}/P_{nm})$. This yields $\pi \in W_0$ and so we can conclude that $\overline{\pi} \in \overline{W}_0$. ☐

3.2 Iterative Construction and Finite Memory

In the proof of Theorem 3.1 we have constructed a winning strategy f for player 1 for the game $\mathcal{G}$ from some position v_0 from a winning strategy $\overline{f}$ for player 1 for the game $\overline{G}_{v_0}$ from $\{v_0\}$. This raises a question. What can we say about the strategy f, if the strategy $\overline{f}$ is simple, that means, can be implemented with small memory? Of course, since the positions of the game $\overline{G}_{v_0}$ are sets of positions of the game G, it is conjecturable that if G is finite, then the size of the memory which is needed is somehow exponential in the number of positions. This is in fact true, but to implement winning strategies with finite memory, we still need an update mechanism which produces the set $v(\pi a v)$ from the set $v(\pi)$ by a certain update rule which depends only on the set $v(\pi)$ and the pair (a, v) but not on the prefix π.

In this section we present such an update rule and we will see how we can use it to implement winning strategies for finite partial information games with finite memory. The idea of this update mechanism has first been used by Reif in [Rei84]. He has used it to define an alternating polynomial space algorithm for the strategy problem for Reif-games.

We will also use this update rule to give an effective iterative construction of the game graph of $\overline{G}_{v_0}$ which can be done in time exponential in $|V|$. The idea of the construction is that for a position $v(\pi)$ and an action a we can construct the set of all a-successors of $v(\pi)$ in the game $\overline{G}_{v_0}$ by iterating over all a-successors v of the set $v(\pi)$ in the game G and computing the set $v(\pi a v)$ using the update rule for each such v. So in particular, the set of a-successors of a position $v(\pi)$ in the game $\overline{G}_{v_0}$ does not depend on the prefix π but only on the set $v(\pi)$ and the action a.

This important observation leads to the idea of a universal game with full information. This means, from a game $\mathcal{G}$ with partial information, we can construct iteratively, a game G^u with full information and positions from 2^V, such that for each $v \in V$ the corresponding game $\overline{G}_v$ with full information is exactly the subgame of G^u which is induced by all the positions that are reachable from $\{v\}$. So in particular we have that for each $v \in V$, player 1 has a winning strategy for $\mathcal{G}$ from v if and only if he has a winning strategy from $\{v\}$ in G^u. Notice that no position from 2^V is duplicated and of course the iterative construction of the game graph of G^u can be done in time exponential in $|V|$ as well.

Similar ideas have been applied in [CDHR06] to *define*, for a game with partial information, a corresponding game with full information. This construction is quite related to our universal powerset construction.

Now consider a game

$$\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A)) \text{ with } G = (V, V_0, (f_a)_{a\in A}, W_0)$$

with partial information. Let $v_0 \in V$ and let $\overline{A}$ be a representative system for $A/{\sim_1^A}$ with $A_1 \subseteq \overline{A}$.
For a set $S \subseteq V$ of positions and a set $B \subseteq A$ of actions we define

- $\mathrm{Post}_B(S) := \{v \in V \mid \exists\, s \in S,\ \exists\, b \in B \ :\ b \in \mathrm{act}(s) \wedge f_b(s) = v\}$ and
- $\mathrm{act}(S) := \bigcap\{\mathrm{act}(s) \mid s \in S\}$.

Furthermore let

- $[v]_{\sim_1} := \{w \in V \mid v \sim_1^V w\}$ for $v \in V$ and
- $[a]_{\sim_1} := \{b \in A \mid a \sim_1^A b\}$ for $a \in A$.

The following fundamental property of the sets $v(\pi)$ for $\pi \in P_{\mathrm{fin}}(v_0)$ gives us the desired update rule, that means, it puts us in the position to give an iterative process that computes these sets step by step.

Proposition 3.6. *Let $\pi \in P_{\mathrm{fin}}(v_0)$, $v \in V$ and $a \in A$ with $\pi a v \in P_{\mathrm{fin}}(v_0)$. Then*

$$v(\pi a v) = \mathrm{Post}_{[a]_{\sim_1}}(v(\pi)) \cap [v]_{\sim_1}.$$

Theorem 3.4. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V)), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a\in A}, (\mathrm{col}, \mathcal{F}_0))$ *be a finite information compatible Muller-game such that* $\mathrm{act}([u]_{\sim_1}) = \mathrm{act}(u)$ *for all* $u \in V_1$*. Then for each* $v_0 \in \mathrm{Win}_1^{\mathcal{G}}$*, player* 1 *has a memory winning strategy for* $\mathcal{G}$ *from* v_0 *which uses at most* $2^{|V|} \cdot m$ *memory states, where* $m = 1$*, if* G *is a parity game and* $m = (|C|)!$*, else. The corresponding memory structure can be constructed in time exponential in* $|V|$*.*

Proof. We define the memory structure $M = (2^V, \delta_0 : \{v_0\} \to \{\{v_0\}\}, \delta)$ by $\delta(S,(a,v)) := \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1}$ for all $(a,v) \in A \times V$ and all $S \in 2^V$. Clearly we can construct M in time exponential in $|V|$.

Now let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in\overline{A}}, (\overline{\mathrm{col}}, \mathcal{F}_0))$ be the corresponding Muller-game with full information. Then player 1 has a winning strategy for $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$ and thus he has a memory winning strategy $g : T \times \overline{V}_1 \to \overline{A}$ for $\overline{G}_{v_0}$ from $\overline{v}_0$ with respect to some memory structure $K = (T, \gamma_0, \gamma)$ where $|T| \leq (|C|)!$ and $|T| = 1$, if $\overline{G}_{v_0}$ is a parity game. We define the memory structure $M^\wedge K := (2^V \times T, \delta_0^\wedge\gamma_0, \delta^\wedge\gamma)$ for $\mathcal{G}$ as follows. First let $\delta_0^\wedge\gamma_0(v_0) := (\delta_0(v_0), \gamma_0(\{v_0\}))$. Now let $(S,t) \in 2^V \times T$ and $(a,v) \in A \times V$. If $\delta(S,(a,v)) \in \overline{V}$ then we define $\delta^\wedge\gamma((S,t),(a,v)) := (\,\delta(S,(a,v)), \gamma(t,(a',\delta(S,(a,v))))\,)$ for the uniquely determined $a' \in \overline{A}$ with $a' \sim_1^A a$. If $\delta(S,(a,v)) \notin \overline{V}$, let $\delta^\wedge\gamma((S,t),(a,v)) := (S,t)$.

Now we define $f : (2^V \times T) \times V_1 \to A$ as follows. Let $((S,t),v) \in (2^V \times T) \times V_1$. If $S \in \overline{V}_1$ and $v \sim_1^V u$ for some $u \in S$ then let $f((S,t),v) := g(t,S)$. Otherwise let $f((S,t),v)$ be an arbitrary action from $\mathrm{act}(v)$ while regarding the condition that f has to be constant over equivalence classes of positions. Then f is a memory strategy for player 1 for $\mathcal{G}$ with respect to $M^\wedge K$.

Using Proposition 3.6, a simple induction over i yields for all $\pi = v_0 a_0 \dots a_{i-1} v_i \in P_{\mathrm{fin}}(v_0)$ the following holds. $(\delta^\wedge\gamma)^*(\pi) = (\overline{v}_i, \gamma^*(\overline{v}_0 a_0' \dots a_{i-1}' \overline{v}_i))$ for the uniquely determined actions $a_0', \dots, a_{i-1}' \in \overline{A}$ with $a_j' \sim_1^A a_j$ for $j = 0, \dots, i-1$, where $\overline{v}_j = v(v_0 a_0 \dots a_{j-1} v_j)$ for $j = 0, \dots, i$. Now similar arguments as in the proof Theorem 3.1 yield that f is a memory winning strategy for player 1 for $\mathcal{G}$ from v_0 with respect to $M^\wedge K$. ☐

Theorem 3.5. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V)), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a\in A}, W_0)$ *be a finite game with partial information and omega-regular winning condition such that* $\mathrm{act}([u]_{\sim_1}) = \mathrm{act}(u)$ *for all* $u \in V_1$*. Then for all* $v_0 \in \mathrm{Win}_1^{\mathcal{G}}$*, player* 1 *has a memory wining strategy for* $\mathcal{G}$ *from* v_0 *which uses only finitely many memory states. The corresponding memory structure and the memory winning strategy can be constructed effectively.*

Proof. Let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{W}_0)$ be the corresponding game with full information and let $\mathcal{A} = (\overline{V}A, Q, q_0, \gamma, \mathrm{col})$ be a deterministic parity automaton with $L(\mathcal{A}) = \overline{W}_0$. We define the game $\overline{G}' = (\overline{V} \times Q, \overline{V}_0 \times Q, (\overline{E}'_a)_{a\in A}, \mathrm{col})$ as follows. For $(\overline{v}, q) \in \overline{V} \times Q$ and $a \in A$, let $(\overline{v}, q)\overline{E}'_a = \{(\overline{w}, \gamma(q, \overline{v}a)) \mid (\overline{v}, \overline{w}) \in \overline{E}_a\}$ and $\mathrm{col}((\overline{v}, q)) = \mathrm{col}(q)$. Then player 1 has a winning strategy for $\overline{G}'$ from $(\overline{v}, q_0)$ and since $\overline{G}'$ is a parity game, player 1 has a positional winning strategy $g : \overline{V}_1 \times Q \to A$ for $\overline{G}'$ from $(\overline{v}, q_0)$. Now we define the memory structure $M = (\overline{V} \times Q, \delta : \{v_0\} \to \{(\{v_0\}, q_0)\}, \delta)$ for $\mathcal{G}$ by $\delta((\overline{v}, q), (a, v)) := (Post_{[a]_{\sim_1}}(\overline{v}) \cap [v]_{\sim_1}, \gamma(q, \overline{v}a))$ for $(\overline{v}, q) \in$

$\overline{V} \times Q$ and $(a, v) \in A \times V$. Furthermore we define $f : (\overline{V} \times Q) \times V_1 \to A$ as follows. Let $((\overline{v}, q), v) \in (\overline{V} \times Q) \times V_1$. If $v \sim_1^V u$ for some $u \in \overline{v}$ then we define $f((\overline{v}, q), v) := g((\overline{v}, q))$. Otherwise we let $f((\overline{v}, q), v)$ be an arbitrary action from $\mathrm{act}(v)$ while regarding the condition that f has to be constant over equivalence classes of positions. Then f is a memory winning strategy for player 1 for $\mathcal{G}$ from v_0 with respect to M. □

Iterative Construction of $\overline{G}_{v_0}$.

We define the sets $V^i \subseteq 2^V$ for $i < \omega$ inductively as follows.

- $V^0 = \{\{v_0\}\}$.
- $V^{i+1} = \{\mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1} \mid S \in V^i,\ a \in A,\ v \in \mathrm{Post}_a(S)\ \}. \cup \{\{v_0\}\}$

Furthermore, for $a \in \overline{A}$ and $i < \omega$, $E_a^i \subseteq 2^V \times 2^V$ is the union of the following sets.

- $E_a^{i,0} = \{(S, \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1}) \mid S \in V^i,\ v \in \mathrm{Post}_{[a]_{\sim_1}}(S),\ S \subseteq V_0\ \}$.
- $E_a^{i,1} = \{(S, \mathrm{Post}_a(S) \cap [v]_{\sim_1}) \mid S \in V^i,\ v \in \mathrm{Post}_a(S),\ a \in \mathrm{act}(S),\ S \subseteq V_1\ \}$.

Remark. Notice that for all $S \subseteq V_1$ we have $\mathrm{Post}_a(S) = \mathrm{Post}_{[a]_{\sim_1}}(S)$.

Proposition 3.7. *For all $i < \omega$ the following propositions hold.*

(1) $V^i = \{v(\pi) \mid \pi \in P_{\mathrm{fin}}(v_0),\ l(\pi) \leq i + 1\}$.
(2) $E_a^i = \{(\overline{v}, \overline{w}) \in \overline{E}_a \mid \overline{v} \in V^i\}$ *for all* $a \in \overline{A}$.

Clearly this yields $\overline{V} = \bigcup\{V^i \mid i < \omega\}$ and thus $\overline{E}_a = \bigcup\{E_a^i \mid i < \omega\}$ for $a \in \overline{A}$. Now if $\mathcal{G}$ is finite, then there is some $\overline{i} \leq 2^{|V|}$ such that $V^{\overline{i}} = V^{\overline{i}+1}$. Obviously this implies $V^{\overline{i}} = V^j$ for all $j > \overline{i}$ and it also yields that for each $a \in \overline{A}$ we have $E_a^{\overline{i}} = E_a^j$ for all $j > \overline{i}$. This yields the following result.

Theorem 3.6. *Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0))$ be a finite information compatible Muller-game and let $v_0 \in V$. Then the corresponding Muller-game $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in \overline{A}}, (\overline{\mathrm{col}}, \mathcal{F}_0))$ with full information can be constructed in time exponential in $|V|$.*

Now using Theorem 3.1, Theorem 3.6, Proposition 2.3 and the corresponding results for games with full information, we obtain the following result.

Theorem 3.7. *The strategy problem for finite information compatible Muller-games can be solved in time exponential in $|V|$.*

Remark. The complexity of the strategy problem for finite games with partial information depends highly on the size of the equivalence classes of positions. The reason is that for a finite game $\mathcal{G}$ with partial information the actual number of positions in the corresponding game with full information is bounded by $k \cdot 2^m \leq |V| \cdot 2^m$, where k is the number of equivalence classes of positions in $\mathcal{G}$ and m is the size of the largest such equivalence class. In particular, the strategy problem for a class of finite information compatible (co-) Büchi games where the size of the equivalence classes of positions is bounded can be solved in time polynomial in $|V|$ and winning strategies can be implemented with polynomially many memory states.

3.2.1 A Universal Powerset Construction

Now we want to use the insight about the powerset construction that we have gained from the results about the iterative construction, to define, for a game $\mathcal{G}$ with partial information, the mentioned universal game G^u with full information.

We define the game in such a way that the set $V^u \subseteq 2^V$ of positions of G^u is downward closed, that means, if $s \in V^u$ and $t \in 2^V$ with $t \subseteq s$, then $t \in V^u$. We will use this property of G^u in Section 3.7.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information. We define the universal game with full information

$$G^u = (V^u, V_0^u, (E_a^u)_{a \in A^u}, W_0^u)$$

as follows, where A^u is a representative system for $A/{\sim_1^A}$ with $A_1 \subseteq A^u$.

- $V^u = \{S \in 2^V \setminus \emptyset \mid \exists v \in V : S \subseteq [v]_{\sim_1} \}$.
- $V_0^u = \{S \in V^u \mid S \subseteq V_0\}$.
- For $a \in A^u$, E_a^u is the union of the following two sets.
    - $E_a^{u,0} := \{(S, \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1}) \mid \ v \in \mathrm{Post}_{[a]_{\sim_1}}(S),\ S \in V_0^u \ \}$
    - $E_a^{u,1} := \{(S, \mathrm{Post}_a(S) \cap [v]_{\sim_1}) \mid \ v \in \mathrm{Post}_a(S),\ a \in \mathrm{act}(S),\ S \in V_1^u \ \}$
- For a play $\pi^u = S_0 a_0 S_1 a_1 \ldots$ in G^u from $S_0 = \{v_0\}$ we define $\pi^u \in W_1^u$ $:\Longleftrightarrow$ for each play $\pi = v_0 a_0' v_1 a_1' \ldots \in P(v_0)$ in G from v_0 with $a_i' \sim_1^A a_i$ and $v_i \in S_i$ for all $i < \omega$ we have $\pi \in W_1$.

Theorem 3.8. *Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ be a game with partial information and let $G^u = (V^u, V_0^u, (E_a^u)_{a \in A^u}, W_0^u)$ be the corresponding universal game with full information. Now let $v \in V$.*

(1) If there is a terminal position $v(\pi) \in V_1^u$ for some $\pi \in P_{\mathrm{fin}}(v)$, then player 1 does not have a strategy for $\mathcal{G}$ from initial position v.

(2) If there is no such terminal position $v(\pi) \in V^u$, then player 1 has a winning strategy for $\mathcal{G}$ from v if and only if he has a winning strategy for G^u from $\{v\}$.

Proof. Let $\overline{G}_v = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A^u}, \overline{W}_0)$ be the corresponding game with full information. Then according to Proposition 3.7 we have $\overline{E}_a = \{(S,T) \in E_a^u \mid S \in \overline{V}\} = E_a^u \cap \overline{V} \times \overline{V}$ for all $a \in A^u$. Thus, each play in G^u from $\overline{v} = \{v\}$ is a play of $\overline{G}_v$ from $\overline{v}$ and vice versa. Furthermore we have $\overline{W}_1 \cap P(\overline{v}) = W_1^u \cap P(\overline{v})$ and so by Theorem 3.1, the proof is finished. □

Theorem 3.9. *Let $\mathcal{G} = (G, (\mathrm{vis}_i^V)), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0)$ be a finite information compatible Muller-game such that $\mathrm{act}([u]_{\sim_1}) = \mathrm{act}(u)$ for all $u \in V_1$. Furthermore let $W \subseteq \mathrm{Win}_1^{\mathcal{G}}$ such that for all $u, v \in W$ with $u \neq v$ we have $u \not\sim_1^V v$. Then player 1 has a memory strategy f for $\mathcal{G}$ from positions in W which uses at most $2^{|V|} \cdot m$ memory states, where $m = 1$, if G is a parity game and $m = (|C|)!$, else, such that f is a winning strategy from each $w \in W$. The corresponding memory structure can be constructed in time exponential in $|V|$.*

3.3 Lower Bounds for the Memory

In this section we prove a lower bound for the amount of memory which is needed to implement winning strategies in games with partial information. The main result is that in general for time bounded safety games with partial information, exponential memory is needed to win. Time bounded safety games are a generalization of safety games. Each time bounded safety winning condition is ω-regular and each time bounded safety game with full information is uniformly positional determined. After we have proved this lower bound, we use the same idea to show that in general for usual safety games at least subexponential memory is needed to win. Furthermore we suggest certain possibilities to strengthen these results.

Definition 3.1. A *time bounded safety game* with time bound $\alpha \leq \omega$ is given by a tuple $G = (V, V_0, (f_a)_{a \in A}, (R, \alpha))$ where $R \subseteq V$, and the winning condition is given by $\pi = v_0 a_0 v_1 a_1 \ldots \in W_0$ if and only if $v_i \in R$ for all $i < \alpha$. If $\alpha = \omega$, then G is a safety game.

The following proposition lists some properties of time bounded safety games which are important for the understanding of our results in this section. Proposition (1) is obvious and proposition (3) can be derived from proposition (2) in the same way as Theorem 3.4 is proved. The proof of proposition (2) is a refinement of the corresponding proof for usual safety games.

Proposition 3.8. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a time bounded safety game with partial information and time bound* α.

- *(1)* W_0 *is* ω*-regular and can be recognized by a deterministic safety automaton with* $\beta + 2$ *states, where* $\beta = \alpha$, *if* $\alpha < \omega$ *and* $\beta = 0$, *otherwise.*
- *(2)* G *is uniformly positional determined.*
- *(3) If* $v \in V$ *such that player* 1 *has a winning strategy for* $\mathcal{G}$ *from* v, *then he has a memory winning strategy for* $\mathcal{G}$ *from* v *which uses at most* $2^{|V|}$ *states.*

Proof. (2) Let $G = (V, V_0, (f_a)_{a \in A}, (R, \alpha))$ be a time bounded safety game and let $Q := V \backslash R$. We define the sets $\mathrm{Attr}_1^i(Q)$ for $i < \omega$ by $\mathrm{Attr}_1^0(Q) = Q$ and $\mathrm{Attr}_1^{i+1}(Q) = \mathrm{Attr}_1^i(Q) \cup \{v \in V_1 \,|\, \exists\ (v,w) \in E : w \in \mathrm{Attr}_1^i(Q)\} \cup \{v \in V_0 \,|\, \forall\ (v,w) \in E : w \in \mathrm{Attr}_1^i(Q)\}$, where E is the edge-relation of the game graph of G. Furthermore, let $\mathrm{Attr}_1^\omega(Q) := \bigcup \{\mathrm{Attr}_1^i(Q) \,|\, i < \omega\}$.

We define positional strategies f for player 1 and g for player 0 as follows. First let $v \in V_1$. If $v \notin \mathrm{Attr}_1^\omega(Q)$ or $v \in Q$, let $f(v)$ be arbitrary. Otherwise let $i := \min\{j < \omega \,|\, v \in \mathrm{Attr}_1^j(Q)\} > 0$. Then there is some $(v,w) \in E$ such that $w \in \mathrm{Attr}_1^{i-1}(Q)$ and we define $f(v) := a$ for some $a \in \mathrm{act}(v)$ with $f_a(v) = w$. Now let $v \in V_0$. If $v \in Q$, let $g(v)$ be arbitrary. If $v \notin \mathrm{Attr}_1^\omega(Q)$, then there is some $(v,w) \in E$, such that $w \notin \mathrm{Attr}_1^\omega(Q)$ and we define $g(v) := a$ for some $a \in \mathrm{act}(v)$ with $f_a(v) = w$. Otherwise, let $i := \max\{j < \omega \,|\, v \notin \mathrm{Attr}_1^j(Q)\} > 0$. Then there is some $(v,w) \in E$ such that $w \notin \mathrm{Attr}_1^{i-1}(Q)$ and we define $g(v) := a$ for some $a \in \mathrm{act}(v)$ with $f_a(v) = w$.

Now we show that f is a winning strategy for player 1 for G from each position $v \in \mathrm{Attr}_1^\beta(Q)$ and g is a winning strategy for player 0 for G from each position

$v \in V \setminus \mathrm{Attr}_1^{\beta}(Q)$, where $\beta = \alpha - 1$, if $\alpha < \omega$ and $\beta = \omega$, otherwise. So consider an arbitrary $v \in V$ and let $\pi = v_0 a_0 v_1 \ldots$ be a play in G from v. If $v \in \mathrm{Attr}_1^{\omega}(Q)$ and π is compatible with f, an easy induction on $i := \min\{j < \omega \mid v \in \mathrm{Attr}_1^{j}(Q)\}$ shows that $v_j \in Q$ for some $j \leq i$. Thus, if $i < \alpha$, then π is won by player 1. Now let $v \in \mathrm{Attr}_1^{\omega}(Q)$ and let π be compatible with g. By induction on $i := \max\{j < \omega \mid v \notin \mathrm{Attr}_1^{j}(Q)\}$ we show that $v_j \notin Q$ for all $j < i$. The cases $i = 0, 1$ are obvious so let $i > 1$. If $v_j \notin \mathrm{Attr}_1^{i-1}(Q)$ for all $j < \omega$, then clearly $v_j \notin Q$ for all $j < \omega$. Otherwise let $l := \min\{j < \omega \mid v_j \in \mathrm{Attr}_1^{i-1}(Q)\} > 0$. Then the play $v_j a_j v_{j+1} \ldots$ is compatible with g and by induction hypothesis, $v_{j+r} \notin Q$ for all $r < i - 1$. So $v_j \notin Q$ for all $j < i - 1 + l$ and $i - 1 + l \geq i$. Thus, if $i \geq \alpha$, then π is won by player 0. Finally let $v \notin \mathrm{Attr}_1(Q)$, let π be compatible with g and assume that π is not won by player 0. Let $i := \min\{j < \omega \mid v_j \in \mathrm{Attr}_1(Q)\} > 0$ and let $l < \omega$ with $v_i \in \mathrm{Attr}_1^{l}(Q)$. Obviously, $v_{i-1} \notin V_0$ by definition of g. On the other hand, if $v_{i-1} \in V_1$, then $(v_{i-1}, v_i) \in E$ yields $v_{i-1} \in \mathit{Attr}_1^{l+1}(Q)$, which is a contradiction to the choice of i. So π is won by player 0. ☐

Now let $1 < n < \omega$. We construct the partial information time bounded safety game

$$\mathcal{G}_n = (G_n, (\mathrm{vis}_{i,n}^{V}), (\mathrm{vis}_{i,n}^{A})) \text{ with } G_n = (V^n, V_0^n, (f_a^n)_{a \in A^n}, (\{r\}, 2n+2))$$

as follows. First let $\{b_1, \ldots, b_m\} \subseteq \{\sigma : N \to N \mid \sigma \text{ is bijective }\}$ be a subset of the set of all permutations of $N := \{1, \ldots, n\}$.

- $V^n = \{v_0\} \uplus \{x_1, \ldots, x_n\} \uplus \{y_1, \ldots, y_n\} \uplus \{r, s\}$
- $\mathrm{vis}_{1,n}^{V}(v_0) = v_0$ and $\mathrm{vis}_{1,n}^{V}(r) = r$ and $\mathrm{vis}_{1,n}^{V}(s) = s$.
- $\mathrm{vis}_{1,n}^{V}(x_i) = x$ for $i \in N$ and $\mathrm{vis}_{1,n}^{V}(y_i) = y$ for $i \in N$
- $V_0^n = \{v_0\} \uplus \{x_1, \ldots, x_n\} \uplus \{r, s\}$
- $A^n = \{a_1, \ldots, a_n\} \uplus \{b_1, \ldots, b_m\} \uplus \{c_1, \ldots, c_n\} \uplus \{d_1, \ldots, d_n\} \uplus \{\circlearrowleft\}$
- $\mathrm{vis}_{1,n}^{A}(a_i) = a$ for $i \in N$ and $\mathrm{vis}_{1,n}^{A}(\circlearrowleft) = \circlearrowleft$
- $\mathrm{vis}_{1,n}^{A}(b_i) = b_i$ for $i \in \{1, \ldots, m\}$, $\mathrm{vis}_{1,n}^{A}(c_i) = c_i$ and $\mathrm{vis}_{1,n}^{A}(d_i) = d_i$ for $i \in N$
- $v_0 \xrightarrow{a_i} x_i$ for $i \in N$ and $r \xrightarrow{\circlearrowleft} r$ and $s \xrightarrow{\circlearrowleft} s$
- $y_i \xrightarrow{d_i} r$ for $i \in N$ and $y_i \xrightarrow{d_j} s$ for $i, j \in N$ with $i \neq j$.
- $y_i \xrightarrow{c_j} x_i$ for $i \in N$ and $j \in N \setminus \{i\}$ and $y_i \xrightarrow{c_i} y_i$ for $i \in N$
- $x_i \xrightarrow{b_j} y_{b_j(i)}$ for $i \in N$ and $j \in \{1, \ldots, m\}$

An example of such a game is depicted in Figure 3.3.

Remark. We could also use more compact games by deleting all the actions d_i for $i \in \{1, \ldots, n\}$ and the position s and instead redirect each action c_i at the position v_i, from v_i itself to the position r. The main result from this section still holds for this case, but the games as we actually use them, are somewhat easier to handle in the proofs.

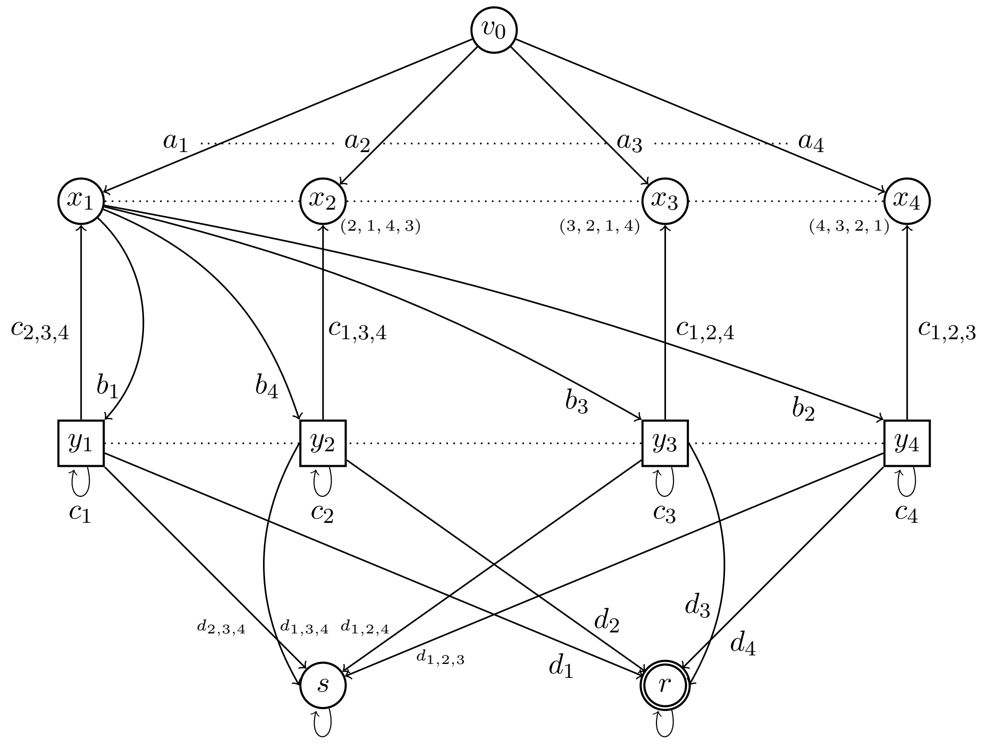

Figure 3.3: The game $\mathcal{G}_4$ where $m = n = 4$ and $b_1 = (1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4) = id_N$, $b_2 = (4, 1, 2, 3)$, $b_3 = (3, 4, 1, 2)$, $b_4 = (2, 3, 4, 1)$.

Proposition 3.9. *Let $n < \omega$. Then player 1 has a memory winning strategy for $\mathcal{G}_n$ from v_0 that uses $2^n - 1$ memory states.*

Proof. We define the memory structure $M = (2^N \setminus \{\emptyset\}, \delta_0, \delta)$ for $\mathcal{G}_n$ as follows. First, $\delta_0(v_0) = N$, $\delta(N, (a_i, x_i)) = N$ and $\delta(N, (b_i, y_j)) = N$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Second, for $i, j \in \{1, \dots, n\}$ and $L \subseteq N$ we define $\delta(L, (c_i, x_j)) = L \setminus \{i\}$, if $|L| > 1$ and $\delta(L, (c_i, x_j)) = L$, if $|L| = 1$. Furthermore, for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and $L \subseteq N$, let $\delta(L, (b_i, y_j)) = b_i(L)$. Finally, $\delta(L, (c_i, y_i)) = \{i\}$ and $\delta(L, (d_i, r)) = \delta(L, (d_i, s)) = N$ for $L \subseteq N$ and $i \in \{1, \dots, n\}$. To complete the picture let $\delta(N, (\circlearrowleft, r)) = N$ and $\delta(N, (\circlearrowleft, s)) = N$. Now we define the memory strategy $f : (2^N \setminus \{\emptyset\}) \times V_1 \to A$ for player 1 for $\mathcal{G}_n$ as follows. For $i \in \{1, \dots, n\}$ and $L \subseteq N$, let $f(L, y_i) := c_j$ for some $j \in L$ if $|L| > 1$ and $f(L, y_i) := d_j$, if $L = \{j\}$ for some $j \in \{1, \dots, n\}$.

Now let $\pi = v_0 a_i \alpha_0 \beta_0 \gamma_0 \zeta_0 \alpha_1 \beta_1 \gamma_1 \zeta_1 \dots$ be a play in $\mathcal{G}_n$ that is compatible with f. For $k < \omega$, let $\pi_k := v_0 a_i \dots \alpha_k \beta_k \gamma_k$ and let $L_k := \delta^*(\pi_k)$. By induction on k we show that for any $k < \omega$, if $\alpha_k \in \{x_1, \dots, x_n\}$ then $|L_k| = n - k$ and $j \in L_k$, where $\gamma_k = y_j$.

For $k = 0$ we have $\alpha_k = \alpha_0 \in \{x_1, \dots, x_n\}$ and $L_k = L_0 = \delta^*(\pi_0) = N$, so there is nothing to show. Now let $k > 0$ such that $\alpha_k \in \{x_1, \dots, x_n\}$. Now assume that there is some $l < k$ such that $\alpha_l \in \{y_1, \dots, y_n\}$ and let l be minimal with this property, that means, $\alpha_r \in \{x_1, \dots, x_n\}$ for all $r < l$. Clearly $l > 0$. Now since $\alpha_{l-1} \in \{x_1, \dots, x_n\}$ we have $\gamma_{l-1} = y_j$ for some $j \in \{1, \dots, n\}$ and so $\alpha_l \in \{y_1, \dots, y_n\}$ yields $\zeta_{l-1} = c_j$. By definition of δ we have $\delta^*(\pi_{l-1}\zeta_{l-1}\alpha_l) = \delta(L_{l-1}, (\zeta_{l-1}, \alpha_l)) = \delta(L_{l-1}, (c_j, y_j)) = \{j\}$. Now π is compatible with f and thus $\beta_l = f(\delta^*(\pi_{l-1}\zeta_{l-1}\alpha_l), \alpha_l)) = f(\{j\}, \alpha_l) = d_j$. But this clearly contradicts $\alpha_k \in \{x_1, \dots, x_n\}$ and thus $\alpha_{k-1} \in \{x_1, \dots, x_n\}$. So the induction hypothesis yields $|L_{k-1}| = n - k + 1$ and $p \in L_{k-1}$ where $p \in \{1, \dots, n\}$ such that $\gamma_{k-1} = y_p$.

Since π is compatible with f we have $\zeta_{k-1} = f(L_{k-1}, \gamma_{k-1})$ and since $\alpha_k \in \{x_1, \dots, x_n\}$ we have $\zeta_{k-1} \in \{c_1, \dots, c_n\}$. So $|L_{k-1}| > 1$ and the definition of f yields $\zeta_{k-1} = c_j$ for some $j \in L_{k-1}$. Thus $\delta^*(\pi_{k-1}\zeta_{k-1}\alpha_k) = \delta(L_{k-1}, (c_j, \alpha_k)) = L_{k-1} \setminus \{j\}$. Now let $l \in \{1, \dots, n\}$ such that $\beta_k = b_l$. Then $|L_k| = |\delta^*(\pi_k)| = |\delta(L_{k-1} \setminus \{j\}, (\beta_k, \gamma_k))| = |b_l(L_{k-1} \setminus \{j\})| = |L_{k-1} \setminus \{j\}| = |L_{k-1}| - 1 = n - k$, since $j \in L_{k-1}$. Furthermore, let $q \in \{1, \dots, n\}$ such that $\gamma_k = y_q$. As we have noticed, $p \in L_{k-1}$ and since $\alpha_k \in \{x_1, \dots, x_n\}$ and $\zeta_{k-1} = c_j$ we have $p \neq j$. So $q = b_l(p) \in b_l(L_{k-1} \setminus \{j\})$.

So there must be some $0 < k \leq n$ such that $\alpha_k \in \{y_1, \dots, y_n\}$ or $\alpha_k \in \{r, s\}$. Now let k be minimal with this property, that means, $\alpha_l \in \{x_1, \dots, x_n\}$ for all $l < k$. Then in particular $\alpha_{k-1} \in \{x_1, \dots, x_n\}$ and so we have $|L_{k-1}| = n - k + 1$ and $j \in L_{k-1}$ where $j \in \{1, \dots, n\}$ such that $\gamma_{k-1} = y_j$. First let $\alpha_k \in \{y_1, \dots, y_n\}$. Then $k < n$. (If $k = n$, then $|L_{k-1}| = 1$, that means, $L_{k-1} = \{j\}$, and since π is compatible with f we have $\zeta_{k-1} = f(\delta^*(\pi_{k-1}), \gamma_{k-1}) = f(L_{k-1}, \gamma_{k-1}) = f(\{j\}, y_j) = d_j$. So $\alpha_k = r$ which is a contradiction to $\alpha_k \in \{y_1, \dots, y_n\}$.) Furthermore, $\zeta_{k-1} = c_j$ and $\alpha_k = y_j$. Thus we have $\delta^*(\pi_{k-1}\zeta_{k-1}\alpha_k) = \delta(L_{k-1}, (\zeta_{k-1}, \alpha_k)) = \delta(L_{k-1}, (c_j, y_j)) = \{j\}$ and since π is compatible with f this yields $\beta_k = f(\delta^*(\pi_{k-1}\zeta_{k-1}\alpha_k), \alpha_k) = f(\{j\}, y_j) = d_j$. So $\gamma_k = f_{d_j}(y_j) = r$ and the number of positions in π up to position α_k is at most $1 + 2k + 1 \leq 2n$. So π is won by player 1. Now let $\alpha_k \in \{r, s\}$. Since π is compatible with f we have $f(L_{k-1}, \gamma_{k-1}) = f(\delta^*(\pi_{k-1}), \gamma_{k-1}) = \zeta_{k-1} \in \{d_1, \dots, d_n\}$ and so the definition of f yields $|L_{k-1}| = 1$. Furthermore, $j \in L_{k-1}$ yields $L_{k-1} = \{j\}$ and so $\zeta_{k-1} = f(L_{k-1}, \gamma_{k-1}) = f(\{j\}, y_j) = d_j$. Thus, $\alpha_k = f_{d_j}(y_j) = r$. Finally, the number of positions in π up to position γ_{k-1} is at most $1 + 2k \leq 2n + 1$ and so, π is won by player 1. $\square$

Now let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, (R, \alpha))$ be an arbitrary information compatible time bounded safety game with partial information, that means, for all $u, v \in V$ with $u \sim_1^V v$ we have $u, v \in R$ or $u, v \notin R$. Then the winning condition $W_1 = \{v_0a_0v_1a_1 \ldots \in P_{\mathrm{fin}}(V) \mid \exists i < \alpha : v_i \in R\ \}$ is observation based. So if $v_0 \in V$ and $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A}, \overline{W}_0)$ is the corresponding game with full information, then $\overline{W}_1 = \{\overline{\pi} \in P_{\mathrm{fin}}(\overline{v}_0) \mid \mathrm{obs}_1(\overline{\pi}) \in \mathrm{obs}_1(W_1)\} = \{\overline{v}_0a_0\overline{v}_1a_1 \ldots \in P_{\mathrm{fin}}(\overline{v}_0) \mid \exists i < \alpha : \overline{v}_i \in \overline{R}\ \}$, where $\overline{v}_0 = \{v_0\}$ and $\overline{R} = 2^R \cap \overline{V}$. So $\overline{G}_{v_0}$ is again a time bounded safety game and the winning condition is given by $(\overline{R}, \alpha)$.

Proposition 3.10. *Let $n < \omega$ and let $\{b_1, \dots, b_m\} = \{\sigma \in N^N \mid \sigma$ is bijective $\}$. Furthermore let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A}, (\overline{R}, 2n+2))$ be the corresponding game with full information for $\mathcal{G}_n$ and let $\overline{f}$ be a winning strategy for player 1 for $\overline{G}_{v_0}$ from $\overline{v}_0 = \{v_0\}$. Then for any $\emptyset \neq L \subseteq N$ there is a finite play prefix $\overline{\pi} \in P_{\mathrm{fin}}(\overline{v}_0)$ such that $\overline{\pi}$ is compatible with $\overline{f}$ and* $\mathrm{last}(\overline{\pi}) = \{y_i \mid i \in L\}$.

Proof. We proceed by induction on $0 \leq n-k < n$ where $k := |L|$. If $n-k = 0$ then $L = N$ and we have $\mathrm{Post}_{[a_1]_{\sim_1}}(\{v_0\}) \cap [x_1]_{\sim_1} = \{x_1,\dots,x_n\}$ and $\mathrm{Post}_{[b_1]_{\sim_1}}(\{x_1,\dots,x_n\}) \cap [y_1]_{\sim_1} = \{y_1,\dots,y_n\}$. Notice that $[b_j]_{\sim_1} = \{b_j\}$ for all $j \in \{1,\dots,m\}$. So $\overline{v}_0 a_1\{x_1,\dots,x_n\}b_1\{y_1,\dots,y_n\}$ is a finite prefix of a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ and for trivial reasons, this prefix is compatible with $\overline{f}$. Now let $0 < n-k < n$. Then $k < n$ and so there is a set $L' \subseteq N$ with $|L'| = k+1$. By induction hypothesis, there is some $\overline{\pi}^0 \in P_{\mathrm{fin}}(\overline{v}_0)$ which is compatible with $\overline{f}$ such that $\mathrm{last}(\overline{\pi}^0) = \{y_i \mid i \in L'\}$. Now assume that $\overline{f}(\overline{\pi}^0) = d_j$ for some $j \in \{1,\dots,n\}$. Since $|L'| > 1$ we have $s \in \mathrm{Post}_{d_j}(\{y_i \mid i \in L'\})$ and so $\overline{\pi}^0 d_j\{s\}\circlearrowleft\{s\}\dots$ is a play in $\overline{G}_{v_0}$ which is compatible with $\overline{f}$ but not won by player 1. This is a contradiction to the fact that $\overline{f}$ is a winning strategy for player 1 for $\overline{G}_{v_0}$ from $\overline{v}_0$. So $\overline{f}(\overline{\pi}^0) = c_{j_0}$ for some $j_0 \in \{1,\dots,n\}$. Now if $j_0 \notin L'$, then $\mathrm{Post}_{c_{j_0}}(\mathrm{last}(\overline{\pi}^0)) \cap [x_1]_{\sim_1} = \{x_i \mid i \in L'\}$ and so $\overline{\pi}^1 = \overline{\pi}^0 c_{j_0}\{x_i \mid i \in L'\}b_1\{y_i \mid i \in L'\}$ is a finite prefix of a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ which is compatible with $\overline{f}$, where w.l.o.g. we assume that $b_1 = \mathrm{id}_N$. Now we can apply the same argumentation as before to show that $\overline{f}(\overline{\pi}^1) = c_{j_1}$ for some $j_1 \in \{1,\dots,n\}$. Again, if $j_1 \notin L'$, then $\overline{\pi}^2 = \overline{\pi}^1 c_{j_1}\{x_i \mid i \in L'\}b_1 \,\mathrm{last}(\overline{\pi}^0)$ is a finite prefix of a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ which is compatible with $\overline{f}$ and so on. But since $\overline{f}$ is a winning strategy for player 1 for $\overline{G}_{v_0}$ from $\overline{v}_0$ there has to be some $k < \omega$ such that $\overline{f}(\overline{\pi}^k) = c_{j_k}$ for some $j_k \in L'$. Furthermore, there is some $l \in \{1,\dots,m\}$ such that $b_l(L' \setminus \{j_k\}) = L$. By construction of $\overline{\pi}^k$ we have $\mathrm{last}(\overline{\pi}^k) = \mathrm{last}(\overline{\pi}^0) = \{y_i \mid i \in L'\}$ and so $\overline{\pi}^k c_{j_k}\,\mathrm{Post}_{c_{j_k}}(\mathrm{last}(\overline{\pi}^0)) \cap [x_1]_{\sim_1} b_l\,\mathrm{Post}_{b_l}(\mathrm{Post}_{c_{j_k}}(\mathrm{last}(\overline{\pi}^0)) \cap [x_1]_{\sim_1}) \cap [y_1]_{\sim_1} = \overline{\pi}^k c_{j_k}\{x_i \mid i \in L' \setminus \{j_k\}\}b_l\{y_i \mid i \in b_l(L' \setminus \{j_k\})\} = \overline{\pi}^k c_{j_k}\{x_i \mid i \in L' \setminus \{j_k\}\}b_l\{y_i \mid i \in L\}$ is a finite prefix of a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ which is compatible with $\overline{f}$. □

Proposition 3.11. *Let $n < \omega$ and let $\{b_1,\dots,b_m\} = \{\sigma : N \to N \mid \sigma \text{ is bijective }\}$. Then each memory winning strategy for player 1 for $\mathcal{G}_n$ from v_0 uses at least $2^n - 1$ memory states.*

Proof. Let $M = (S, \delta_0, \delta)$ be a memory structure for $\mathcal{G}_n$ with $|S| < 2^n - 1$ and assume that there is a memory winning strategy $f : S \times V_1 \to A$ for player 1 for $\mathcal{G}_n$ from v_0 with respect to M. We construct the memory structure $\overline{M} = (S, \overline{\delta}_0, \overline{\delta})$ for $\overline{G}_{v_0}$ with $\overline{\delta}_0 : \{v_0\} \to \{\delta_0(v_0)\}$ and $\overline{\delta} : S \times (A \times \overline{V}) \to S$ as follows. For $t \in S$ and $(a, \overline{v}) \in A \times V$ we define $\overline{\delta}(t,(a,\overline{v})) := \delta(t,(a,v))$ for some $v \in \overline{v}$. Notice that δ is constant over the set $\{(t,(a,v')) \mid v' \in \overline{v}\}$ and so this definition is independent of the chosen v. Now we define the memory strategy $\overline{f} : S \times \overline{V}_1 \to A$ for player 1 for $\overline{G}_{v_0}$ with respect to $\overline{M}$ by $\overline{f}(t, \overline{v}) := f(s, y_1)$ for $t \in S$ and $\overline{v} \in \overline{V}_1$. Notice that f is constant over the set $\{(t, y_i) \mid i = 1,\dots,n\}$ and so this definition yields the same function for any y_i, $i \in \{1,\dots,n\}$. Since $\overline{f}$ depends only on the memory state t and not on the position $\overline{v}$, we regard $\overline{f}$ as a function $S \to A$. Now consider the corresponding game $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A}, (\overline{R}, 2n+2))$ with full information for $\mathcal{G}_n$. With similar arguments as in the proof of Theorem 3.1 one can show that $\overline{f}$ is a memory winning strategy for player 1 for $\overline{G}_{v_0}$ from $\overline{v}_0$. Now we shall see that this is not possible.

According to Proposition 3.10, for any $\emptyset \neq L \subseteq N$ there is a finite play prefix $\overline{\pi} \in P_{\mathrm{fin}}(\overline{v}_0)$ such that $\overline{\pi}$ is compatible with $\overline{f}$ and $\mathrm{last}(\overline{\pi}) = \{y_i \mid i \in L\}$. So there are two prefixes $\overline{\pi}_1, \overline{\pi}_2 \in P_{\mathrm{fin}}(\overline{v}_0)$ with $\mathrm{last}(\overline{\pi}_1), \mathrm{last}(\overline{\pi}_2) \subseteq \{y_1,\dots,y_n\}$ and $\mathrm{last}(\overline{\pi}_1) \neq$

$\mathrm{last}(\overline{\pi}_2)$ such that $\overline{\pi}_1$ and $\overline{\pi}_2$ are compatible with $\overline{f}$ and $\overline{\delta}^*(\overline{\pi}_1) = \overline{\delta}^*(\overline{\pi}_2)$. Now we consider extensions of $\overline{\pi}_1$ and $\overline{\pi}_2$ to prefixes $\overline{\pi}_l^i \in P_{\mathrm{fin}}(\overline{v}_0)$, $i < \omega$, $l = 1, 2$ with $\overline{\pi}_l^0 = \overline{\pi}_l$ such that the following holds. First, $\overline{\pi}_l^i$ is compatible with $\overline{f}$ and player 0 always chooses id_N when it is his turn. Furthermore, $\overline{\pi}_l^{i+1}$ extends $\overline{\pi}_l^i$ by exactly two moves. Finally, nondeterministic moves are resolved by choosing the maximal successor set if an edge from $\overline{E}_{c_j}$ has to be chosen and by choosing $\{s\}$, if an edge from $\overline{E}_{d_j}$ has to be chosen, where $j \in \{1, \ldots, n\}$.

Now let $i < \omega$ with $\mathrm{last}(\overline{\pi}_l^i) \subseteq \{y_1, \ldots, y_n\}$ for $l = 1, 2$ such that $\mathrm{last}(\overline{\pi}_1^i) \neq \mathrm{last}(\overline{\pi}_2^i)$ and $t := \overline{\delta}^*(\overline{\pi}_1^i) = \overline{\delta}^*(\overline{\pi}_2^i)$. Then $\overline{f}(t) = c_j$ for some $j \in \{1, \ldots, n\}$, since otherwise, for at least one $l \in \{1, 2\}$ there would be an extension of $\overline{\pi}_l^i$ to a play in $\overline{G}_{v_0}$ from $\overline{v}_0$ which is compatible with $\overline{f}$ but not won by player 1. Now if $y_j \in \mathrm{last}(\overline{\pi}_1^i) \cap \mathrm{last}(\overline{\pi}_2^i)$, $|\mathrm{last}(\overline{\pi}_1^i)| > 1$ and $|\mathrm{last}(\overline{\pi}_1^i)| > 1$ then $\overline{\delta}^*(\overline{\pi}_1^{i+1}) = \overline{\delta}^*(\overline{\pi}_2^{i+1})$ and $|\mathrm{last}(\overline{\pi}_1^{i+1}) \cap \mathrm{last}(\overline{\pi}_2^{i+1})| < |\mathrm{last}(\overline{\pi}_1^i) \cap \mathrm{last}(\overline{\pi}_2^i)|$. So we can conclude that there must be some $i < \omega$ with $\mathrm{last}(\overline{\pi}_j^i) \subseteq \{y_1, \ldots, y_n\}$ for $j = 1, 2$ and $t := \overline{\delta}^*(\overline{\pi}_1^i) = \overline{\delta}^*(\overline{\pi}_2^i)$, such that $\overline{f}(t) = c_j$ for some $j \in \{1, \ldots, n\}$ and $y_j \notin \mathrm{last}(\overline{\pi}_l^i)$ or $|\mathrm{last}(\overline{\pi}_l^i)| = 1$ for some $l \in \{1, 2\}$.

But now consider an arbitrary extension $\overline{\pi}$ of $\overline{\pi}_l^{i+1}$ to a play in $\overline{G}_{v_0}$ from $\overline{v}$ which is compatible with $\overline{f}$. By construction of $\overline{\pi}_l^{i+1}$, some set $\overline{v} \subseteq \{y_1, \ldots, y_n\}$ is visited twice during $\overline{\pi}$ and so the first $2n + 2$ position of $\overline{\pi}$ cannot belong to the set $\{\{r\}\}$. (Notice that if $\overline{\pi}$ reaches the set $\{\{r\}\}$, then for any $k \leq n$, at least two positions of size k have to be seen in $\overline{\pi}$.) Thus, $\overline{\pi}$ is lost by player 1. ☐

Theorem 3.10. *There is a sequence* $(\mathcal{G}_n)_{n<\omega}$ *of time bounded safety games with partial information and some designated initial position* v_0*, such that for each* $n < \omega$*, the number of positions in* $\mathcal{G}_n$ *and the time bound are linear in* n *and the following holds. Player* 1 *has a memory winning strategy for* $\mathcal{G}_n$ *from* v_0 *which uses* $2^n - 1$ *memory states but he does not have such a strategy which uses at most* $2^n - 2$ *memory states.*

Now there are two unsatisfying points in this result. First we have considered *time bounded* safety games where a serious (that means finite) time bound is imposed on the game. And second, the number of actions in each game $\mathcal{G}_n$ is exponential in n. Now although this is not yet proved, it seems very reasonable that even if we delete the time bound from the game $\mathcal{G}_n$, there is no memory winning strategy for player 1 for $\mathcal{G}_n$ from v_0 which uses at most $2^n - 2$ memory states. We formulate this as a conjecture and then, with the same idea as for time bounded safety games, we show that for safety games without a time bound, at least subexponential memory is needed to win in general. Furthermore we suggest sets $\{b_1, \ldots, b_m\}$ with $m = n$ for which it would be interesting to know the exact amount of memory which is needed to win in the corresponding games (with or without time bound).

Conjecture 3.1. *For* $1 < n < \omega$ *let* $\mathcal{G}_n$ *be the game as defined above without a time bound and let* $\{b_1, \ldots, b_m\} = \{\sigma : N \to N \mid \sigma \text{ is bijective }\}$*. Then each memory winning strategy for player* 1 *for* $\mathcal{G}_n$ *from* v_0 *uses at least* $2^n - 1$ *memory states.*

Now let $1 < n < \omega$. We construct the partial information safety game

$$\mathcal{G}_n = (G_n, (\mathrm{vis}_{i,n}^V), (\mathrm{vis}_{i,n}^A)) \text{ with } G_n = (V^n, V_0^n, (f_a^n)_{a \in A^n}, \{r\})$$

as follows. First let $\{b_1, \dots, b_m\} \subseteq \{\sigma : N \to N \mid \sigma \text{ is bijective }\}$ be a subset of the set of all permutations of $N := \{1, \dots, n\}$.

- $V^n = \{v_0\} \uplus \{x_1, \dots, x_n\} \times N \uplus \{y_1, \dots, y_n\} \times N \uplus \{r, s\}$
- $\text{vis}^V_{1,n}(v_0) = v_0$ and $\text{vis}^V_{1,n}(r) = r$ and $\text{vis}^V_{1,n}(s) = s$.
- $\text{vis}^V_{1,n}((x_i, k)) = (x, k)$ and $\text{vis}^V_{1,n}((y_i, k)) = (y, k)$ for $i, k \in N$
- $V^n_0 = \{v_0\} \uplus \{x_1, \dots, x_n\} \times N \uplus \{r, s\}$
- $A^n = \{a_1, \dots, a_n\} \uplus \{b_1, \dots, b_m\} \uplus \{c_1, \dots, c_n\} \uplus \{d_1, \dots, d_n\} \uplus \{\circlearrowright\}$
- $\text{vis}^A_{1,n}(a_i) = a$ for $i \in N$ and $\text{vis}^A_{1,n}(\circlearrowright) = \circlearrowright$
- $\text{vis}^A_{1,n}(b_i) = b_i$ for $i \in \{1, \dots, m\}$, $\text{vis}^A_{1,n}(c_i) = c_i$ and $\text{vis}^A_{1,n}(d_i) = d_i$ for $i \in \{1, \dots, n\}$
- $v_0 \overset{a_i}{\to} (x_i, 1)$ and $r \overset{\circlearrowright}{\to} r$ and $s \overset{\circlearrowright}{\to} s$
- $(y_i, n) \overset{d_i}{\to} r$ for $i \in N$ and $(y_i, n) \overset{d_j}{\to} s$ for $i, j \in N$ with $i \neq j$
- $(y_i, k) \overset{c_j}{\to} (x_i, k+1)$ for $i \in N$, $k \in N \setminus \{n\}$ and $j \in N \setminus \{i\}$
- $(y_i, k) \overset{c_i}{\to} (y_i, k)$ for $i \in N$ and $k \in N \setminus \{n\}$.
- $(x_i, k) \xrightarrow{b_j} (y_{b_j(i)}, k)$ for $i, k \in N$ and $j \in \{1, \dots, m\}$

Now with very similar arguments as for the case of time bounded safety games one can show that for each $1 < n < \omega$, player 1 has a memory winning strategy for $\mathcal{G}_n$ from v_0 which uses $2^n - 1$ memory states but he does not have a memory winning strategy for $\mathcal{G}_n$ from v_0 which uses at most $2^n - 2$ memory states. Notice that the number of positions in $\mathcal{G}_n$ is now *quadratic* in n. Thus we have established a *subexponential* lower bound for the memory which is needed in general to win in safety games with partial information.

Theorem 3.11. *There is a sequence $(\mathcal{G}_n)_{n<\omega}$ of safety games with partial information and some designated initial position v_0, such that for each $n < \omega$, the number of positions in $\mathcal{G}_n$ is quadratic in n and the following holds. Player 1 has a memory winning strategy for $\mathcal{G}_n$ from v_0 which uses $2^n - 1$ memory states but he does not have such a strategy which uses at most $2^n - 2$ memory states.*

Proposition 3.12. *Let $1 < n < \omega$. Then there are permutations $\sigma_i \in S_n$ for $i = 1, \dots, n$ such that for all $i, j \in \{1, \dots, n\}$ with $i \neq j$ we have $\sigma_i(k) \neq \sigma_j(k)$ for all $k \in \{1, \dots, n\}$.*

Proof. We define the $n \times n$-matrix $A \in \{1, \dots, n\}^{n \times n}$ as follows. $A_{i,i} = 1$ for $i \in \{1, \dots, n\}$, $A_{i,j+l} = l + 1$ for $i, j, l \in \{1, \dots, n\}$ such that $j + l \leq n$ and $A_{i+l,j} = n - (l - 1)$ for $i, j, l \in \{1, \dots, n\}$ such that $i + l \leq n$. Clearly, each column and each row of A contains any number $i \in \{1, \dots, n\}$ exactly once. Thus the rows $A_{i,-}$, $i = 1, \dots, n$ of A define permutations σ_i, $i = 1, \dots, n$ for which the proposition holds. $\square$

Now for $0 < n < \omega$ consider the game $\mathcal{G}_n$ with $\{b_1, \ldots, b_m\} = \{\sigma_1, \ldots, \sigma_n\}$. Since the number of actions in $\mathcal{G}_n$ is linear in n, it would be very interesting to figure out the exact amount of memory which is needed to implement a winning strategy for player 1 from v_0 in the most efficient way, both for the case where there is a finite time bound and for the case where no finite time bound is imposed.

3.4 Hiding Private Moves

In this section we consider the strategy problem in the case where private moves are hidden. We use a modified version of the powerset construction for the case where private moves are not hidden to solve this problem. We do not prove the results which are analog to the results for the case where private moves are not hidden since the proofs are just adaptions of the proofs in this case. They are more technical but no new methods are applied.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a game with partial information, let $v_0 \in V$ and let $i \in \{0, 1\}$. We define the corresponding game

$$\tilde{G}^i_{v_0} = (\tilde{V}, \tilde{V}_0, (\tilde{E}_a)_{a\in A}, \tilde{W}_0)$$

with full information as follows. To simplify the notation we let $i = 1$ and we denote the game $\tilde{G}^i_{v_0}$ by $\tilde{G}_{v_0}$.

- $\tilde{V} = \{v^+(\pi) \mid \pi \in P_{\mathrm{fin}}(v_0)\}$.
- $v^+(\pi) = \{\mathrm{last}(\pi') \mid \pi' \in P_{\mathrm{fin}}(v_0),\ \pi \sim_1^+ \pi'\}, \quad$ for $\pi \in P_{\mathrm{fin}}(v_0)$.
- $\tilde{V}_0 = \{v^+(\pi) \in \tilde{V} \mid \mathrm{last}(\pi) \in V_0\}$.
- For $a \in A$, the edge relation $\tilde{E}_a$ is the union of
  $\tilde{E}^0_a := \{(v^+(\pi), v^+(\pi b v)) \mid b \sim_1^A a,\ v^+(\pi) \in \tilde{V}_0\}$ and
  $\tilde{E}^1_a := \{(v^+(\pi), v^+(\pi a v)) \mid a \in \bigcap_{v\in v^+(\pi)} act(v),\ v^+(\pi) \in \tilde{V}_1\}$.
- For a play $\tilde{\pi} = \tilde{v}_0 a_0 \tilde{v}_1 a_1 \ldots \in P(\tilde{v}_0)$ in $\tilde{G}_{v_0}$ from $\tilde{v}_0 = v^+(v_0)$ we define $\tilde{\pi} \in \tilde{W}_1$ $:\Longleftrightarrow$
  for each play $\pi = v_0 a'_0 v_1 a'_1 \ldots \in P(v_0)$ in G from v_0 such that there are numbers $0 =: k_0 < k_1 < k_2 < \ldots$ with $v_{k_i}, \ldots, v_{k_{i+1}-1} \in \tilde{v}_i$ and $a'_{k_{i+1}-1} \sim_1^A a_i$ for all $i < \omega$ and $k_{i+1} - k_i = 1$, if $\tilde{v}_i \in \tilde{V}_1$, we have $\pi \in W_1$.

Again, we fix some representative system $\tilde{A}$ for $A/\sim_1^A$ such that $A_1 \subseteq \tilde{A}$ and we let the actions of $\tilde{G}_{v_0}$ come from $\tilde{A}$.

Proposition 3.13. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a\in A}, W_0)$ *be a game with partial information, let* $v_0 \in V$ *and let* $\tilde{G}_{v_0} = (\tilde{V}, \tilde{V}_0, (\tilde{E}_a)_{a\in\tilde{A}}, \tilde{W}_0)$ *be the corresponding game with full information. Furthermore let* $\tilde{v}_0 = v^+(v_0)$.

(1) For $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ *with* $v^+(\pi) = v^+(\pi')$ *we have* $v^+(\pi a w) = v^+(\pi' a w)$ *for all* $a \in A$ *and all* $w \in V$ *such that* $\pi a w, \pi' a w \in P_{\mathrm{fin}}(v_0)$.

(2) *If* $\pi \in P_{\mathrm{fin}}(v_0)$ *and* $(v^+(\pi), \tilde{w}) \in \tilde{E}_a$ *then* $(v^+(\pi), \tilde{w}) = (v^+(\pi'), v^+(\pi' b w))$ *for some* $\pi' \in P_{\mathrm{fin}}(v_0)$*, some* $w \in V$ *and some* $b \in A$ *such that* $b \sim_1^A a$ *and* $\pi' \sim_1^+ \pi$*.*

(3) *If* $(\tilde{v}, \tilde{w}) \in \tilde{E}_a$ *with* $\tilde{v} \neq \tilde{w}$ *if* $\tilde{v} \in \tilde{V}_0$*, then for all* $w \in \tilde{w}$ *there is a finite play prefix* $v_1 a_1 v_2 a_2 \dots a_{k-1} v_k$ *such that* $v_k = w$*,* $v_i \in \tilde{w}$ *for* $i = 2, \dots, k$*,* $v_1 \in \tilde{v}$ *and* $a \sim_1^A a_1$*. If* $\tilde{v} \in \tilde{V}_1$*, then* $a_1 = a$*. If* $\tilde{w} \in \tilde{V}_1$ *then we can choose* $k = 2$*.*

(4) *For each finite prefix* $\tilde{\pi} = \tilde{v}_0 a_0 \dots a_{n-1} \tilde{v}_n \in P_{\mathrm{fin}}(\tilde{v}_0)$ *with* $\tilde{v}_i \neq \tilde{v}_{i+1}$ *if* $\tilde{v}_i \in \tilde{V}_0$ *and all* $v \in \tilde{v}_n$ *there is a finite prefix* $\pi = v_0 a'_0 \dots a'_{m-1} v_m \in P_{\mathrm{fin}}(v_0)$ *with* $v_m = v$ *such that there are numbers* $0 =: k_0 < k_1 < \dots < k_{n+1} = m + 1$ *with* $v_{k_i}, \dots, v_{k_{i+1}-1} \in \tilde{v}_i$ *and* $a'_{k_{i+1}-1} \sim_1^A a_i$ *for all* $0 \leq i < n$ *and* $k_{i+1} - k_i = 1$*, if* $\tilde{v}_i \in \tilde{V}_1$*.*
Moreover, for each such prefix, $v(v_0 a'_0 \dots a'_{k_{i+1}-2} v_{k_{i+1}-1}) = \tilde{v}_i$ *for all* i*.*

Theorem 3.12. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a game with partial information, let* $v_0 \in V$ *and let* $\tilde{G}_{v_0} = (\tilde{V}, \tilde{V}_0, (\tilde{E}_a)_{a \in \tilde{A}}, \tilde{W}_0)$ *be the corresponding game with full information for the case where private moves are hidden.*

(1) *If there is a terminal position* $\tilde{v} \in \tilde{V}_1$*, then player* 1 *does not have a strategy for* $\mathcal{G}$ *from initial position* v_0*, if private moves are hidden.*

(2) *If there is no terminal position* $\tilde{v} \in \tilde{V}_1$*, then player* 1 *has a winning strategy for* $\mathcal{G}$ *from* v_0 *if private moves are hidden if and only if he has a wining strategy for* $\tilde{G}_{v_0}$ *from* $\tilde{v}_0 = v^+(v_0)$*.*

3.4.1 Strongly Observation Based Winning Conditions

In this section we consider observation based winning conditions for the case where private moves are hidden, which we call strongly observation based. We show that from each game $\mathcal{G}$ with partial information, where the winning condition is strongly observation based, we can construct a game $\mathcal{G}'$ with partial information, where the winning condition is observation based, such that for each position v, player 1 has a winning strategy for $\mathcal{G}$ from v if private moves are hidden, if and only if, he has a winning strategy for $\mathcal{G}'$ from v.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information, let $v_0 \in V$ and let $\tilde{G}_{v_0} = (\tilde{V}, \tilde{V}_0, (\tilde{E}_a)_{a \in \tilde{A}}, \tilde{W}_0)$ be the corresponding game with full information.

For a sequence $\pi = v_0 a_0 v_1 a_1 \dots \in V(AV)^* \cup V(AV)^\omega$, let $\overleftarrow{\pi}$ be the sequence from $V(AV)^* \cup V(AV)^\omega$ which is obtained from π by contracting each maximal sequence $v_0 a_0 v_1 a_1 \dots a_n v_{n+1} (\dots)$ of private moves of player 0 in π to v_0. Now we call $\mathrm{obs}_1^+(\pi) = \mathrm{obs}_1(\overleftarrow{\pi})$ the sequence of observations of player 1 in π if private moves are hidden. (For the definition of obs_1, see Section 3.1.1.) We call the winning condition W_1 strongly observation based, if for all $\pi, \pi' \in P(V)$ with $obs_1^+(\pi) = obs_1^+(\pi')$ we have $\pi \in W_1$ if and only if $\pi' \in W_1$. Then $\pi \in W_1$ if and only if $\mathrm{obs}_1^+(\pi) \in \mathrm{obs}_1^+(W_1) := \{\mathrm{obs}_1^+(\pi) \mid \pi \in W_1\}$.

Notice that in the case of a strongly observation based winning condition, the private moves of player 0 are not relevant to a play. Player 0 just makes those moves to reach some position, but the particular sequence of private moves which player 0 chooses to reach the position, does not affect the winner of the play. Thus, we can eliminate all the private moves of player 0 from the game and in return give him the possibility to go immediately to any position, which is reachable by a sequence, consisting of private moves of player 0 and one move of player 0 which is not private.

Proposition 3.14. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a game with partial information such that* W_1 *is strongly observation based. Then there is a game* $\mathcal{G}' = (G', (\mathrm{vis}_i^V), (\mathrm{vis}_1^A))$ *with* $G' = (V, V_0, (f'_a)_{a \in A'}, W'_0)$ *such that* W'_1 *is observation based and for each* $v_0 \in V$*, player* 1 *has a winning strategy for* $\mathcal{G}$ *from* v_0 *if private moves are hidden if and only if he has a winning strategy for* $\mathcal{G}'$ *from* v_0*.*

Proof. First we construct the nondeterministic game $G' = (V, V_0, (E'_a)_{a \in A}, W'_0)$ as follows. Let $E = \bigcup\{E_a \mid a \in A\}$ be the edge relation of the underlying game graph of G, let $E_{\mathrm{pr},0} := \{(v,w) \in E \mid \ v \in V_0,\ v \sim_1^V w\ \}$ be the set of all private moves of player 0 in $\mathcal{G}$ and for $a \in A$ let E''_a be the set of all pairs (v,w) such that there is some finite play prefix $v = w_0 a_0 w_1 a_1 \dots a_{n-2} w_{n-1} a_{n-1} w_n = w$ in G with $(w_i, w_{i+1}) \in E_{\mathrm{pr},0}$ for $i = 0, \dots, n-2$, $(w_{n-1}, w_n) \notin E_{\mathrm{pr},0}$ and $a_{n-1} = a$. Furthermore, let $E'_a := (E_a \cup E''_a) \setminus E_{\mathrm{pr},0}$ for $a \in A$. Finally, for each position $v \in V_0$ such that from v there is some infinite sequence of private moves of player 0 in G, we add (v,v) to E'_a for some $a \in \mathit{act}(v)$. The winning condition is defined by $W'_1 = \{\pi \in P^{G'}(V) \mid \mathrm{obs}_1^+(\pi) \in \mathrm{obs}^+(W_1)\}$. Since $\mathrm{obs}_1(\pi) = \mathrm{obs}_1(\pi')$ implies $\mathrm{obs}_1^+(\pi) = \mathrm{obs}_1^+(\pi')$ for all $\pi, \pi' \in P^{G'}(V)$, W'_1 is observation based.

Now from a prefix π of a play in G' we construct the prefix $\overrightarrow{\pi}$ of a play in G by replacing each move $(v,w) \in E'_a \setminus E$ for some $a \in A$ in π in the following way. If (v,w) is not a selfloop of player 0, then we replace the move by some finite play prefix $v = w_0 a_0 w_1 a_1 \dots a_{n-1} w_n = w$ in G with $(w_i, w_{i+1}) \in E_{\mathrm{pr},0}$ for $i = 0, \dots, n-2$, $(w_{n-1}, w_n) \notin E_{\mathrm{pr},0}$ and $a_{n-1} = a$. If (v,w) is a selfloop of player 0 then we distinguish two cases. If π is infinite and all the following positions in π coincide with v, then we replace (v,w) by an infinite sequence of private moves of player 0 in G. If π is finite or one of the following positions in π does not coincide with v, then we just delete the move (v,w) from π. Notice that $\overleftrightarrow{\pi} := \overleftarrow{\overrightarrow{\pi}} = \overleftarrow{\pi}$.

Now let $v_0 \in V$ and let first $f : \{\pi \in P^G_{\mathrm{fin}}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ be a winning strategy for player 1 for $\mathcal{G}$ from initial position v_0 if private moves are hidden. We define $g : \{\pi \in P^{G'}_{\mathrm{fin}}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ by $g(\pi) := f(\overrightarrow{\pi})$ for $\pi \in \mathrm{dom}(g)$. Since f is a strategy if private moves are hidden this definition is independent of the sequences that we have chosen in the definition of $\overrightarrow{\pi}$. Furthermore, if $\pi, \pi' \in \mathrm{dom}(g)$ with $\pi \sim_1^* \pi'$, then and easy induction on $l(\pi)$ yields $\overrightarrow{\pi} \sim_1^+ \overrightarrow{\pi}'$ and so $g(\pi) = f(\overrightarrow{\pi}) = f(\overrightarrow{\pi}') = g(\pi')$. Thus, g is a strategy for player 1 for $\mathcal{G}'$ from v_0.

Now if π is a play in G' from initial position v_0 that is compatible with g then by definition of g, $\overrightarrow{\pi}$ is a play in G from initial position v_0 that is compatible with f and so $\overrightarrow{\pi}$ is won by player 1 in G. (For a finite prefix π' of $\overrightarrow{\pi}$ with $\mathrm{last}(\pi') \in V_1$ consider the uniquely determined finite prefix σ of π such that the number of moves of player 1 in σ is the same as in π'. Furthermore let a be the action in $\overrightarrow{\pi}$ after π' and let b the action in π after σ. Then $\overrightarrow{\sigma} = \pi'$ and $a = b$ and by definition of g we have $f(\pi') = g(\sigma) = b = a$.) This yields $\mathrm{obs}_1^+(\pi) = \mathrm{obs}_1(\overleftarrow{\pi}) = \mathrm{obs}_1(\overleftrightarrow{\pi}) =$

$\mathrm{obs}_1^+(\overrightarrow{\pi}) \in \mathrm{obs}^+(W_1)$ and thus $\pi \in W_1'$. So g is a winning strategy for player 1 for $\mathcal{G}'$ from v_0.

Now let conversely $f : \{\pi \in P_{\mathrm{fin}}^{G'}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ be a winning strategy for player 1 for $\mathcal{G}'$ from initial position v_0. We define $g : \{\pi \in P_{\mathrm{fin}}^{G}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ by $g(\pi) := f(\overleftarrow{\pi})$ for $\pi \in \mathrm{dom}(g)$. If $\pi, \pi' \in \mathrm{dom}(g)$ with $\pi \sim_1^+ \pi'$, then $\overleftarrow{\pi} \sim_1^* \overleftarrow{\pi}'$ (cf. Section 2.2). So we have $g(\pi) = f(\overleftarrow{\pi}) = f(\overleftarrow{\pi}') = g(\pi')$. Thus, g is a strategy for player 1 for $\mathcal{G}$ from v_0 if private moves are hidden.

Now if $\pi = v_0 a_0 v_1 a_1 \ldots$ is a play in G from initial position v_0 that is compatible with g, then we distinguish two cases. If each sequence of private moves of player 0 in π is finite, then by definition of g, $\overleftarrow{\pi}$ is a play in G' from v_0 which is compatible with f and thus, $\overleftarrow{\pi}$ is won by player 1. So $\mathit{obs}_1^+(\pi) = \mathit{obs}_1(\overleftarrow{\pi}) = \mathrm{obs}_1^+(\overleftarrow{\pi}) \in \mathit{obs}^+(W_1)$ and since W_1 is strongly observation based, this yields $\pi \in W_1$. If there is some $i < \omega$ such that $v_j \to v_{j+1}$ is a private move of player 0 for all $j \geq i$, then we consider the minimal such i and we choose some $a \in A$ such that $(v_i, v_i) \in E_a'$. Now let $\sigma := \pi(\leq i)$. By definition of g, $\overleftarrow{\sigma}$ is compatible with f and so, $\pi' := \overleftarrow{\sigma} v_i a v_i a \ldots$ is a play in G' from v_0 which is compatible with f and thus, π' is won by player 1. Therefore, $\mathrm{obs}_1^+(\pi) = \mathrm{obs}_1(\overleftarrow{\pi}) = \mathrm{obs}_1(\overleftarrow{\sigma}) = \mathrm{obs}_1^+(\overleftarrow{\sigma}) = \mathrm{obs}_1^+(\pi') \in \mathrm{obs}_1^+(W_1)$. So $\pi \in W_1$.

Now we transform G' into a deterministic game as follows. We define $A' = A_1 \cup A_0 \times V$ and for $(a, w) \in A_0 \times V$ we define $\mathrm{dom}(f_{(a,w)}) = \{v \in V_0 \mid (v, w) \in E_a'\}$ and $f_{(a,w)}(v) = w$ for all $v \in \mathrm{dom}(f_{(a,w)})$. Furthermore we define $\mathrm{vis}_1^A((a, w)) = \mathrm{vis}_1^A(a)$ for all $(a, w) \in A_0 \times V$. The winning condition is defined as before. Clearly this construction preserves the fact that for each $v_0 \in V$ player 1 has a winning strategy for $\mathcal{G}$ from v_0 if private moves are hidden if and only if he has a winning strategy for $\mathcal{G}'$ from v_0. ☐

Now let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, (\mathrm{col}, \mathcal{F}_0))$ be an information compatible Muller-game. Then for all $\pi \in P^G(V)$ we have $\inf_{\mathrm{col}}(\pi) = \inf_{\mathrm{col}}(\mathrm{obs}_1^+(\pi))$, so the winning condition $W_1 = \{\pi \in P(V) \mid \inf_{\mathrm{col}}(\pi) \notin \mathcal{F}_0\}$ is obviously observation based. Now let $\mathcal{G}'$ be the game from the proof of Proposition 3.14. It is easy to see that $W_1' = \{\pi \in P^{G'}(V) \mid \inf_{\mathrm{col}}(\pi) \notin \mathcal{F}_0\}$ and thus, $\mathcal{G}'$ is again an information compatible Muller-game and the construction preserves parity, (co-) Büchi and (co-) reachability conditions as well.

Furthermore, if $\mathcal{G}$ is finite, then the construction can be done in time polynomial in the size of G. So the construction yields a polynomial time reduction of the strategy problem for information compatible Muller-games if private moves are hidden to the corresponding strategy problem.

Theorem 3.13. *Let* $\mathcal{G} = (G, (\mathrm{vis}_i^V)), (\mathrm{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a finite information compatible Muller-game such that* $\mathrm{act}([u]_{\sim_1}) = \mathrm{act}(u)$ *for all* $u \in V_1$. *Then for each* $v_0 \in \mathrm{Win}_1^{\mathcal{G},h}$, *player* 1 *has a memory winning strategy for* $\mathcal{G}$ *from* v_0 *if private moves are hidden, which uses at most* $2^{|V|} \cdot m$ *memory states, where* $m = 1$, *if* G *is a parity game and* $m = (|C|)!$, *else. The corresponding memory structure can be constructed in time exponential in* $|V|$.

Proof. Let the game $\mathcal{G}'$ with partial information and the relation $E_{pr,0}$ be defined as in the proof of Proposition 3.14. Furthermore let $M = (2^V \times T, \delta_0^\wedge \gamma_0, \delta^\wedge \gamma)$ be the memory structure for $\mathcal{G}'$ from the proof of Theorem 3.4 and let $f : (2^V \times T) \times V_1 \rightarrow A$ be the corresponding memory winning strategy for player 1 for $\mathcal{G}'$ from v_0 with respect to M.

Now we define the memory structure $M' = (2^V \times S, \delta_0^\wedge\gamma_0, \delta')$ as follows. For $(S, t) \in 2^V \times T$ and $(a, v) \in A \times V$ let $\delta'((S, t), (a, v)) = \delta^\wedge\gamma((S, t), (a, v))$, if $S \subseteq V_1$ or $v \not\sim_1^V w$ for $w \in S$ and $\delta'((S, t), (a, v)) = (S, t)$, if $S \subseteq V_0$ and $v \sim_1^V w$ for $w \in S$.

First, an easy induction over n yields that for each $\pi = v_0a_0 \dots a_{n-1}v_n \in P_{fin}^G(v_0)$ we have $(\delta')^*(\pi) = (\delta^\wedge\gamma)^*(\overleftarrow{\pi})$.

Clearly, M' is a memory structure for player 1 for $\mathcal{G}$, if private moves are hidden. Now if $\pi \in P^G(v_0)$ is a play in G from v_0 which is compatible with f with respect to M', then by definition of M', $\overleftarrow{\pi}$ of a play in G' from v_0 which is compatible with f with respect to M. Thus, $\overleftarrow{\pi}$ is won by player 1, that means, $\inf_{\mathrm{col}}(\overleftarrow{\pi}) \notin \mathcal{F}_0$. Now since $\mathcal{G}$ is information compatible we have $\inf_{\mathrm{col}}(\pi) = \inf_{\mathrm{col}}(\overleftarrow{\pi})$ and so, π is won by player 1 as well. So f is a memory winning strategy for player 1 for $\mathcal{G}$ from v_0 if private moves are hidden, with respect to M'. $\square$

3.4.2 Omega-Regular Winning Conditions

In this section we study games with partial information where private moves are hidden and where the winning condition is ω-regular. We construct from a nondeterministic Büchi-automaton recognizing this winning condition a nondeterministic Büchi-automaton recognizing the winning condition of the corresponding game with full information. Of course, from this automaton we also obtain an S1S-formula defining this winning condition but it is not clear how to construct such a formula (from a formula defining the winning condition of the game with partial information) without a translation into automata and vice versa.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a finite game with partial information and let $\mathcal{A} = (VA, Q, q_0, \Delta, F)$ be a Büchi-automaton recognizing W_0. Furthermore let $v_0 \in V$ and let $\tilde{G}_{v_0} = (\tilde{V}, \tilde{V}_0, (\tilde{E}_a)_{a \in A}, \tilde{W}_0)$ be the corresponding game with full information. Then for each play $\tilde{\pi} = \tilde{v}_0a_0\tilde{v}_1a_1 \dots \in P(\tilde{v}_0)$ in $\tilde{G}_{v_0}$ from $\tilde{v}_0 = v^+(v_0)$ we have $\tilde{\pi} \in \tilde{W}_0$ if and only if there is a play $\pi = v_0a_0'v_1a_1' \dots \in W_0$ such that there are numbers $0 =: k_0 < k_1 < k_2 < \dots$ with $v_{k_i}, \dots, v_{k_{i+1}-1} \in \tilde{v}_i$ and $a'_{k_{i+1}-1} \sim_1^A a_i$ for all $i < \omega$ and $k_{i+1} - k_i = 1$, if $\tilde{v}_i \in \tilde{V}_1$.

We define $\mathcal{B} = (\tilde{V}A, \tilde{Q}, (q_0, v_0, 0), \tilde{\Delta}, \tilde{F})$ as follows.

- $\tilde{Q} = Q \times V \times \{0, 1\}$.

- $((p, v, i), \tilde{v}a, (q, w, 0)) \in \tilde{\Delta} :\Longleftrightarrow$
  $v \in \tilde{v}$ and there is a finite play prefix $v_1a_1v_2a_2 \dots a_{n-1}v_n$ in G with $v_1 = v$, $v_j \in \tilde{v}$ for all $1 \leq j \leq n$ and $n = 1$, if $\tilde{v} \in \tilde{V}_1$ and some $b \sim_1^A a$ with $b \in act(v_n)$ and $f_b(v_n) = w$ such that there are $q_1, \dots, q_{n-1} \in Q$ with (p, v_1a_1, q_1), $\dots$, $(q_{n-2}, v_{n-1}a_{n-1}, q_{n-1})$, $(q_{n-1}, v_nb, q) \in \Delta$.

- $((p, v, i), \tilde{v}a, (q, w, 1)) \in \tilde{\Delta} :\Longleftrightarrow$
  $v \in \tilde{v}$ and there is a finite play prefix $v_1a_1v_2a_2 \dots a_{n-1}v_n$ in G with $v_1 = v$, $v_j \in \tilde{v}$ for all $1 \leq j \leq n$ and $n = 1$, if $\tilde{v} \in \tilde{V}_1$ and some $b \sim_1^A a$ with $b \in act(v_n)$

and $f_b(v_n) = w$ such that there are $q_1, \dots, q_{n-1} \in Q$ with (p, v_1a_1, q_1), …, $(q_{n-2}, v_{n-1}a_{n-1}, q_{n-1})$, $(q_{n-1}, v_nb, q) \in \Delta$ and $q_j \in F$ for some $0 < j < n$.

- $\tilde{F} = \{(q, v, i) \mid q \in F\} \cup \{(q, v, 1) \mid q \in Q\}$.

Proposition 3.15. *For each play $\tilde{\pi} = \tilde{v}_0a_0\tilde{v}_1a_1 \ldots \in P(\tilde{v}_0)$ in $\tilde{G}_{v_0}$ from $\tilde{v}_0 = v^+(v_0)$ we have $\tilde{\pi} \in \tilde{W}_0$ if and only if $\tilde{\pi} \in L(\mathcal{B})$.*

Now in order to show that the strategy problem for ω-regular games with partial information where private moves are hidden is decidable, we have to show that we can construct the automaton $\mathcal{B}$ effectively, that means, the two criteria for the membership of a transition to $\tilde{\Delta}$ are decidable.

Proposition 3.16. *The problem to determine whether $t \in \tilde{\Delta}$ for a given transition $t = ((p, v, i), \tilde{v}a, (q, w, j))$ is decidable.*

Proof. First consider the case $j = 0$. If $\tilde{v} \in \tilde{V}_1$, then $t \in \tilde{\Delta}$ if and only if $v \in \tilde{v}$ and there some $b \sim_1^A a$ with $b \in \mathrm{act}(v)$ and $f_b(v) = w$ such that $(p, vb, q) \in \Delta$. Clearly this is decidable. If $\tilde{v} \in \tilde{V}_0$, then $t \in \tilde{\Delta}$ if and only if $v \in \tilde{v}$ and there is a finite play prefix $v_1a_1v_2a_2 \ldots a_{n-1}v_n$ in G with $v_1 = v$, $v_j \in \tilde{v}$ for $j = 1, \dots, n$ and some $b \sim_1^A a$ with $b \in act(v_n)$ and $f_b(v_n) = w$ such that there are $q_1, \dots, q_{n-1} \in Q$ with $(p, v_1a_1, q_1), \dots, (q_{n-1}, v_nb, q) \in \Delta$. Now consider the following nondeterministic finite automaton $\mathcal{C}_0 := (\tilde{v}A, Q \times \tilde{v} \cup \{q\}, (p, v), \Delta_{\mathcal{C}_0}, \{q\})$.

- $((p_1, v_1), v'a', (p_2, v_2)) \in \Delta_{\mathcal{C}_0} :\Longleftrightarrow$
  $v' = v_1$, $a' \in \mathrm{act}(v_1)$, $f_{a'}(v_1) = v_2$ and $(p_1, v'a', p_2) \in \Delta$.
- $((p_1, v_1), v'a', q) \in \Delta_{\mathcal{C}_0} :\Longleftrightarrow$
  $v' = v_1$, $a' \in \mathrm{act}(v_1)$, $f_{a'}(v_1) = w$, $a' \sim_1^A a$ and $(p_1, v'a', q) \in \Delta$.

It is easy to see that $t \in \tilde{\Delta}$ if and only if $L(\mathcal{C}_0) \neq \emptyset$. Since the latter is decidable and the construction of $\mathcal{C}_0$ is clearly effective, the former is decidable as well.

Now let $j = 1$. If $\tilde{v} \in \tilde{V}_1$ then $t \notin \tilde{\Delta}$, so let $\tilde{v} \in \tilde{V}_0$. Then $t \in \tilde{\Delta}$ if and only if $v \in \tilde{v}$ and there is a finite play prefix $v_1a_1v_2a_2 \ldots a_{n-1}v_n$ in G with $v_1 = v$, $v_j \in \tilde{v}$ for $j = 1, \dots, n$ and some $b \sim_1^A a$ with $b \in act(v_n)$ and $f_b(v_n) = w$ such that there are $q_1, \dots, q_{n-1} \in Q$ with $(p, v_1a_1, q_1), \dots, (q_{n-1}, v_nb, q) \in \Delta$ and $q_j \in F$ for some $0 < j < n$. Now consider the following nondeterministic finite automaton $\mathcal{C}_1 := (\tilde{v}A, Q \times \tilde{v} \times \{0, 1\} \cup \{q\}, (p, v, 0), \Delta_{\mathcal{C}_1}, \{q\})$.

- $((p_1, v_1, i), v'a', (p_2, v_2, i)) \in \Delta_{\mathcal{C}_0} :\Longleftrightarrow$
  $v' = v_1$, $a' \in \mathrm{act}(v_1)$, $f_{a'}(v_1) = v_2$ and $(p_1, v'a', p_2) \in \Delta$.
- $((p_1, v_1, 0), v'a', (p_2, v_2, 1)) \in \Delta_{\mathcal{C}_0} :\Longleftrightarrow$
  $v' = v_1$, $a' \in \mathrm{act}(v_1)$, $f_{a'}(v_1) = v_2$, $p_2 \in F$ and $(p_1, v'a', p_2) \in \Delta$.
- $((p_1, v_1, 1), v'a', q) \in \Delta_{\mathcal{C}_0} :\Longleftrightarrow$
  $v' = v_1$, $a' \in \mathrm{act}(v_1)$, $f_{a'}(v_1) = w$, $a' \sim_1^A a$ and $(p_1, v'a', q) \in \Delta$.

It is easy to see that $t \in \tilde{\Delta}$ if and only if $L(\mathcal{C}_1) \neq \emptyset$. Since the latter is decidable and the construction of $\mathcal{C}_1$ is clearly effective, the former is decidable as well. $\square$

3.4.3 Iterative Construction and Finite Memory

Now we want to see that we can construct the game graph of $\tilde{G}_{v_0}$ iteratively in a very similar way as for the case where private moves are not hidden. We formulate the corresponding update rule in the following proposition. First we need some notation. Let $\mathcal{G} = (G, (\text{vis}_i^V), (\text{vis}_i^A))$, $G = (V, V_0, (f_a)_{a \in A}, W_0)$ be a game with partial information, let $v_0 \in V$ and let $\tilde{G}_{v_0}$ be the corresponding game with full information. For a set $S \subseteq V$, let $\text{Reach}^+(S) \supseteq S$ be the set of all positions which are reachable from some position in S by a sequence of private moves of player 0.

Proposition 3.17. *Let* $\pi \in P_{\text{fin}}(v_0)$*, let* $a \in act(\text{last}(\pi))$ *and let* $v = f_a(\text{last}(\pi))$*.*

(1) If $\text{last}(\pi) \in V_0$ *and* $\text{last}(\pi) \sim_1^V v$ *then* $v^+(\pi a v) = v^+(\pi)$*.*
(2) If $\text{last}(\pi) \in V_1$ *or* $\text{last}(\pi) \not\sim_1^V v$ *then* $v^+(\pi a v) = \text{Reach}^+(Post_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1})$*.*

Theorem 3.14. *Let* $\mathcal{G} = (G, (\text{vis}_i^V)), (\text{vis}_i^A))$ *with* $G = (V, V_0, (f_a)_{a \in A}, W_0)$ *be a finite game with partial information and omega-regular winning condition such that* $\text{act}([u]_{\sim_1}) = \text{act}(u)$ *for all* $u \in V_1$*. Then for all* $v_0 \in \text{Win}_1^{\mathcal{G},h}$*, player* 1 *has a memory wining strategy from* v_0 *for* $\mathcal{G}$ *if private moves are hidden which uses only finitely many memory states. The memory structure and the memory winning strategy can be constructed effectively.*

Iterative Construction of $\tilde{G}_{v_0}$.

We define the sets $V^i \subseteq 2^V$ for $i < \omega$ inductively as follows.

- $V^0 = \{\text{Reach}^+(\{v_0\})\}$.
- $$V^{i+1} = \{\text{Reach}^+(\text{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1}) \mid S \in V^i,\ a \in A,\ v \in \text{Post}_a(S) \text{ and } S \subseteq V_1 \text{ or } v \not\sim_1^V u \text{ for all } u \in S\ \} \cup \{\text{Reach}^+(\{v_0\})\}.$$

Furthermore, for $a \in \tilde{A}$ and $i < \omega$, $E_a^i \subseteq 2^V \times 2^V$ is the union of the following sets.

- $E_a^{i,0} = \{(S, \text{Reach}^+(\text{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1})) \mid S \in V^i \cap 2^{V_0},\ v \in \text{Post}_{[a]_{\sim_1}}(S)$ and $S \subseteq V_1$ or $v \not\sim_1^V u$ for all $u \in S\ \}$.
- $E_a^{i,\circlearrowleft} = \{(S, S) \mid S \in V^i \cap 2^{V_0},\ \exists\, b \sim_1^A a,\ \exists\, (u, v) \in E_b : u, v \in S\ \}$.
- $E_a^{i,1} = \{(S, \text{Reach}^+(\text{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1})) \mid S \in V^i \cap 2^{V_1},\ v \in \text{Post}_{[a]_{\sim_1}}(S),$ $a \in \text{act}(S)$ and $S \subseteq V_1$ or $v \not\sim_1^V u$ for all $u \in S\ \}$.

Proposition 3.18. *For all* $i < \omega$ *the following propositions hold.*

(1) $V^i = \{v^+(\pi) \mid \pi \in P_{\text{fin}}(v_0),\ l(\text{vis}_1^+(\pi)) \leq i + 1\ \}$*.*
(2) $E_a^i = \{(\tilde{v}, \tilde{w}) \in \tilde{E}_a \mid \tilde{v} \in V^i\}$ *for all* $a \in \tilde{A}$*.*

3.5 Lower Bounds for the Complexity

Our main goal in this section is to show that the strategy problem for Reif-games is EXPTIME-hard. (For the notion of a Reif-game see Section 2.8.) So together with the fact that the problem can be solved in exponential time this shows that the problem is complete for the class EXPTIME. We use that APSPACE = EXPTIME, so we are dealing with alternating Turing-machines. For the definition of an alternating Turing-machine, the notion of time and space bounds and the computation game of an alternating Turing-machine see for example [WW86].

To prove the EXPTIME-hardness of the strategy problem for Reif-games we need two results about alternating Turing-machines first. The proof of the first Lemma can be found in [CKS81], the proof the second Lemma is an easy adaption of the tape reduction theorem for deterministic Turing-machines, see for example [WW86].

Lemma 3.1. APSPACE = EXPTIME.

Lemma 3.2. *Let $S(n) \geq n$. Then for all $L \in \mathrm{ASPACE}(S(n))$ there is an alternating Turing machine with only one tape and space bound $S(n)$ which accepts L.*

Now we can prove the desired hardness result. The idea for the proof is taken from [Rei84] where a sketch of the original proof by Reif can be found.

Theorem 3.15. *The strategy problem for Reif-games where private moves are hidden is EXPTIME-hard.*

Proof. Let $L \in EXPTIME$. Then according to Lemma 3.1 there is some $k \geq 1$ such that $L \in ASPACE(n^k)$ and so Lemma 3.2 yields an alternating Turing-machine $M = (Q, \Gamma, \Sigma \supseteq \Gamma, q_0, \delta)$ with only one tape and space bound n^k that accepts L.

W.l.o.g. we can assume that there is some $b \in \mathbb{N} \setminus \{0\}$ such that $|\operatorname{Next}(C)| = b$ for each non-final configuration C of M, where $\operatorname{Next}(C)$ is the set of all successor configurations of C. (For b we can use the finite upper bound of the numbers $|\operatorname{Next}(C)|$ where C is some configuration of M. If there are non-final configurations C of M with $|\operatorname{Next}(C)| < b$ then with at most $b-1$ additional final states we can add branches from non-final configurations to final configurations to force the computation trees to be strictly b-braching as desired.) We write $\operatorname{Next}_i(C)$ $i = 1, \ldots, b$ for the b successor configurations of configuration C. Now if $\Gamma = \Sigma \uplus (Q \times \Sigma) \uplus \{\#\}$, then we can describe each configuration C of M by a word

$$\underline{C} = \# w_0 \ldots w_{i-1} (q w_i) w_{i+1} \ldots w_t \# \in \Gamma^*.$$

Since M has space bound n^k and we have $k \geq 1$, w.l.o.g. we can assume that $|\underline{C}| = n^k + 2$ for all configurations C of M on inputs of length n. (We just use a representation of the tape which has a priori the maximum length that will occur during a computation on an input of length n.)

Now for a configuration C of M and some $1 \leq i \leq n^k$ the symbol number i of the word $\underline{C_j}$ where $C_j = \operatorname{Next}_j(C)$ only depends on the symbols number $i-1$, i and $i+1$ of $\underline{C}$. So for each $j \in \{1, \ldots, b\}$ there is a function $f_j : \Gamma^3 \to \Gamma$ such that the following holds. If C is a configuration of M and $1 \leq i \leq n^k$ and the symbols number $i-1$, i and $i+1$ of the string representation $\underline{C}$ of C are $a_{-1}a_0a_1$ then the symbol number i of the string representation $\underline{C_j}$ of the successor configuration $C_j = \text{Next}_j(C)$ of C is $f_j(a_{-1}a_0a_1)$.

Now we define the following alternating algorithm A. For the presentation of the algorithm we abbreviate $c := \underline{C_0(x)}$.

```
input $x \in \Gamma_{in}^*$
$i := -1$, $j := -1$, $initial := true$
while (true)
    universal step : choose $i \in \{1, \dots, n^k\}$
    for ($l = 0, \dots, n^k + 1$)
        if ($l = 0$ or $l = n^k + 1$) : $\gamma := \#$
        else : existential step : guess a symbol $\gamma \in \Gamma$
        if ($\gamma \in Q \times \Sigma$) : $s := \gamma$
        if ($initial = true$ and $\gamma \neq c_l$) : reject
        if ($l = i - 1$) : $a_{-1} := \gamma$
        if ($l = i$) : $a_0 := \gamma$
        if ($l = i + 1$) : $a_1 := \gamma$
        if ($l = j$ and $f_r(a_{-1}a_0a_1) \neq \gamma$) : reject
    endfor
    $j := i$,   $initial := false$
    if ($pr_1(s) \in Q_{acc}$) : accept
    if ($pr_1(s) \in Q_{rej}$) : reject
    if ($pr_1(s) \in Q_\forall$) : universal step : choose $r \in \{1, \dots, b\}$
    if ($pr_1(s) \in Q_\exists$) : existential step : guess $r \in \{1, \dots, b\}$
endwhile.
```

We assume that A has three work tapes and only one single universal state. On the first tape the universal player may store all the other states that he needs, so the head on the first tape remains on the first position of the tape all the time. On the second tape i, j and a_{-1}, a_0, a_1 are stored. All other information is stored on the third tape and in the set of states.

For a configuration $C = (q, \gamma, w_2, w_3, p_0, 0, p_2, p_3, x)$ of A on input x we define $p_0(C) := (\gamma, w_2, p_2)$, $p_1(C) := 0$ and $c(C) := (q, w_3, p_0, p_3)$.
Now let $V := \{(0, p_0(C), c(C), p_1(C)) \mid C$ is a universal configuration of A on x $\} \cup \{(1, p_0(C), c(C), p_1(C)) \mid C$ is an existential configuration of A on x $\}$ and let the move relation E be defined by $(i, p_0(C), c(C), p_1(C)) \to (i', p_0(C'), c(C'), p_1(C'))$ if and only if $C' \in \text{Next}_A(C)$.

Now consider the game $G = (V, E)$. The conditions (R1) and (R2) are trivial for player $\forall$ since he has full information. For player $\exists$ the conditions (R1) and (R2) hold by the definition of G and so G is a Reif-game. Of course G is essentially the computation game $G(A, x)$ of A on input x. We just use a different description of the positions so that the game has the format of a Reif-game. We will identify G with $G(A, x)$ but of course we have to keep in mind the decomposition of the positions.

Furthermore A has a space bound of $O(\log(n^k)) = O(\log(n))$, that means, there is a $d > 0$ such that A has a space bound of $d \cdot \log(n)$. This yields that the number of different configurations of A on input x is bounded by $m \cdot (n+1) \cdot (d \cdot \log(n))^2 \cdot |\Xi|^{2 \cdot d \cdot \log(n)+1} = 2^{O(\log(n))} = n^{O(1)}$ where m is the number of states of A and Ξ is the work alphabet of A. So G has at most $n^{O(1)}$ positions and each such position can be represented by a word of length $O(\log(n))$. (Notice that we do not store the input word in the positions of G.) So we can construct the game G in time $n^{O(1)}$, that means, in time polynomial in $|x|$.

Now we have to show that for each input x of M, player $\exists$ has a winning strategy for the game $G(M,x)$ from initial position $C_0(M,x)$ if and only if player $\exists$ has a winning strategy for the game $G(A,x)$.

So let $x \in \Gamma^*$ and let first $g : \mathrm{dom}(g) \to \{1,\ldots,b\}$ be a positional winning strategy for player $\exists$ for the computation game $G(M,x)$ of M on x from initial position $C_0(M,x)$. From this strategy we construct a winning strategy h for player $\exists$ for G from initial position $C_0(A,x)$ if private moves are hidden.

Note that since such a strategy depends on the history of a play, in our definition of the strategy we can refer to the last configuration that player $\exists$ has chosen. (Of course we cannot store any configuration explicitly in a variable due to the desired space bound.)

Now in pass number l of the inner loop where initial is true, player $\exists$ chooses the symbol number l of the word representation $\underline{C_0(M,x)}$ of the initial configuration $C_0(M,x)$ of M on x. Each time when the inner loop has terminated, if the last configuration C of M that he has chosen was an existential one, player $\exists$ chooses $r = g(C)$. In pass number l of the inner loop where initial is false, if the last configuration that he has chosen was C, player $\exists$ chooses the symbol number l of the word representation $\underline{\mathrm{Next}_r(C)}$ of the successor configuration number r of C.

The value of this strategy depends only on the values of the variable initial, the values of the previously chosen symbols γ and the value of r. Since these variables are all public, the value of the strategy depends only on the part of the history that is visible to player $\exists$, even if private moves are hidden. So h is a strategy for player $\exists$ for the computation game $G(A,x)$ of A on x from initial position $C_0(A,x)$ if private moves are hidden.

Due to the definition of the strategy, if player $\exists$ plays according to this strategy, then he correctly guesses a sequence of successive configurations of M on x, beginning with the initial configuration $C_0(M,x)$. So of course player $\forall$ will never be able to detect any failure by means of the functions f_r, $r = 1,\ldots,b$. Furthermore independently of the numbers r that player $\forall$ chooses when he has to choose, the sequence which player $\exists$ guesses is a finite play of the computation game $G(M,x)$ of M on x from initial position $C_0(M,x)$ which is compatible with g. So this play is won by player $\exists$, that means, it ends with an accepting configuration. Therefore, algorithm A will accept after the last configuration of this play has been guessed by player $\exists$ completely.

Now let conversely g be a winning strategy for player $\exists$ for the computation game $G(A,x)$ of A on x from initial position $C_0(A,x)$ if private moves are hidden.

The strategy g yields a strategy $h : \mathrm{dom}(h) \to \{1,\ldots,b\}$ for player $\exists$ for the computation game $G(M,x)$ of M on x as follows. For a sequence $C_0C_1\ldots C_s \in \mathrm{dom}(h)$ (that means, C_i is a configuration of M on input x for $i = 0,\ldots,s$, $C_0 =$

$C_0(M, x)$, C_s is existential and C_{i+1} is a successor configuration of C_i for $i = 0, \dots, s-1$) we consider a corresponding computation of A on x where player $\exists$ successively chooses the configurations $C_0, C_1, \dots, C_s$ and where the appropriate value for variable r is chosen in each pass of the outer loop.

(That means, in pass number i of the outer loop, player $\exists$ chooses the configuration C_i by successively guessing the appropriate symbols $\gamma = (c_i)_l$ for $l = 1, \dots, n^k$ and afterwards, the player that has to choose the value of the variable r chooses $r = \rho$ where C_{i+1} is the successor configuration number ρ of C_i. Of course such a computation exists since $C_0 C_1 \dots C_s$ is a correct sequence of successive configurations of M on x.)

Since C_s is an existential configuration, the strategy g yields a value ρ for r which player $\exists$ has to choose. We define $h(C_0 \dots C_s) := \rho$.

(Note that this definition is independent of the values i that player $\forall$ chooses in the corresponding computation since the strategy g only depends on the part of a history that is visible to player $\exists$. So for all computations that we can choose to define $h(C_0 \dots C_s)$ the strategy g yields the same value.)

Now let $\pi = C_0 C_1 \dots C_t$ be an arbitrary play of $G(M, x)$ from initial position $C_0 = C_0(M, x)$ that is compatible with h. We have to show that player $\exists$ wins π, that means, C_t is an accepting configuration. For this purpose let π' be a corresponding play of $G(A, x)$, that means, a play such that player $\exists$ successively chooses the configurations $C_0 C_1 \dots C_t$ and such that the appropriate value for variable r is chosen in each pass of the outer loop. (Just as in the definition of h.) We have to show that π' is compatible with g.

We proceed by contradiction, so we assume that π' is not compatible with g. Then there is some $m > 1$ such that up to position $m-1$ (which belongs to player $\exists$) the play π' is compatible with g and position m of the play is not chosen according to g.

Now let $\overline{\pi}$ be obtained from π' by changing all the positions of π' from m onward according to g (and some arbitrary strategy for player $\forall$), so that $\overline{\pi}$ is compatible with g. (When we change some position of π' which is a non-terminal position into a terminal position, then this is the last position of $\overline{\pi}$.)

Since the number of the successor configuration of each existential configuration C_i is defined via g we can conclude that position number m of π' results from a move of player $\exists$ where he guesses some symbol γ^*.

(After successively choosing $C_0, \dots, C_s$ where C_s is existential, player $\exists$ chooses the value ρ for variable r where C_{s+1} is successor configuration number ρ of C_s. Since the play π is compatible with h, we have $h(C_0 \dots C_s) = \rho$ and by definition of h this is exactly the value that g yields on each corresponding computation of A on x where player $\exists$ successively chooses the configurations $C_0, \dots, C_s$ and where the appropriate value for variable r is chosen in each pass of the outer loop. So choosing ρ is compatible with g.)

Now say we changed γ^* into $\gamma^+ \neq \gamma^*$. (That means, in $\overline{\pi}$ player $\exists$ now guesses γ^+ instead of γ^* when he moves to position m of $\overline{\pi}$.)

If then position number m of π' results from some move of player $\exists$ in pass number 0 of the outer loop, then γ^* is the symbol number l of $\underline{C_0}$ for some $l \in \{1, \dots, n^k\}$. Since $\gamma^+ \neq \gamma^*$, in the play $\overline{\pi}$ the test $\gamma \neq c_l$ will return true in pass number l of the inner loop in pass number 0 of the outer loop and thus player $\exists$ loses $\overline{\pi}$, which is a contradiction.

So we conclude that position number m of π' results from some move of player $\exists$ in pass number k of the outer loop for some $k > 0$. So γ^* is the symbol number l of $\underline{C_k}$ for some $l \in \{1, \ldots, n^k\}$. But now consider the play $\tilde{\pi}$ which coincides with $\overline{\pi}$ except for player $\forall$ choosing l in pass number $k-1$ of the outer loop instead of the originally chosen i.

Because g is a strategy in the Reif-game $G(A, x)$ where private moves are hidden, it is independent of this choice and so, since $\overline{\pi}$ is compatible with g, $\tilde{\pi}$ is compatible with g as well. But now let a_{-1}, a_0 and a_1 be the symbols number $l-1$, l and $l+1$ of $\underline{C_{k-1}}$. We know that γ^* is the symbol number l of $\underline{C_k}$ where C_k is the successor configuration number r of the configuration C_{k-1} for some $r \in \{1, \ldots, b\}$.

Furthermore in the play $\tilde{\pi}$ the value r is chosen in pass number $k-1$ of the outer loop and we have $\gamma^+ \neq \gamma^*$. So in the play $\tilde{\pi}$ the test $f_r(a_{-1}a_0a_1) \neq \gamma$ returns true in pass number l of the inner loop in pass number k of the outer loop and thus player $\exists$ loses $\tilde{\pi}$ which is a contradiction as well. Altogether we have shown that the play π' of $G(A, x)$ is compatible with g which concludes the proof. □

Corollary 3.2. *(1) The strategy problem for Reif- games in the case where private moves are hidden is* EXPTIME- *complete.*
(2) The strategy problem for Reif-games is EXPTIME- *complete.*

Proof. We know that the strategy problem for Reif-games where private moves are hidden is EXPTIME-hard and we know that the strategy problem for Reif-games is in EXPTIME. Furthermore it can easily be shown that the construction from Proposition 3.14 (disregarding the winning condition) yields a polynomial time reduction of the strategy problem for Reif-games where private moves are hidden to the strategy problem for Reif-games. □

Corollary 3.3. *(1) The strategy problem for information compatible safety games where private moves are hidden is* EXPTIME- *complete.*
(2) The strategy problem for information compatible safety games with partial information is EXPTIME- *complete.*

3.5.1 Blindfold Games

A blindfold game is a game with partial information where one of the player is blindfolded, that means, he does not get any information about the moves of his opponent, except for the fact that they have happened. (Notice that of course a blindfolded player still considers only such moves of his opponent possible which are compatible with the rules of the games. This is due to the fact that the game graph is common knowledge. But a blindfolded player has no information, which of these possible moves his opponent has actually chosen.) Since we are asking for winning strategies for player 1, we consider games where player 1 is blindfolded.

Definition 3.2. A game $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$, $G = (V, V_0, (f_a)_{a \in A}, W_0)$ with partial information is called a *blindfold game*, if for all $S \subseteq V_0$ with $S \subseteq [u]_{\sim_1}$ for some $u \in V$, the following two conditions hold.

(B1) $\mathrm{Post}_A(S) \subseteq [v]_{\sim_1}$ for some $v \in V$.
(B2) For all $a, b \in \bigcup\{\mathrm{act}(v) \mid v \in S\}$ we have $a \sim_1^V b$.

Now notice that if we apply the powerset construction to a blindfold game, then each of the positions of player 0 has exactly one successor position. Of course there may be several actions leading to this position. But if we consider finite games with observation based winning condition, then the winning condition of the corresponding games with full information does not depend on these actions. So we can expect that the strategy problem for such games is easier to solve. We shall now see that if $\sim_1^A$ implies $\sim_1^V$ (cf. Section 2.5), then the strategy problem for information compatible blindfold parity games is in PSPACE which substantiates this intuition. This result is an extension of Reif's result from [Rei84] which says that the strategy problem for blindfold Reif-games is PSPACE-complete. For the proof we need the Theorem of Savitch which says that nondeterministic Turing-machines which use only polynomial space can be simulated by deterministic Turing-machines which use only polynomial space.

Lemma 3.3. (Theorem of Savitch, [Sav70]) NPSPACE = PSPACE.

Theorem 3.16. *Let $\mathcal{K}$ be a class of information compatible blindfold parity games such that $\sim_1^A$ implies $\sim_1^V$. Then the strategy problem for $\mathcal{K}$ is in* PSPACE*.*

Proof. Let $\mathcal{G} \in \mathcal{K}$, let $v_0 \in V$ and let $\overline{G}_{v_0} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{\mathrm{col}})$ be the corresponding game with full information. We have already noticed that $|\overline{v}\overline{E}| = 1$ for all $\overline{v} \in \overline{V}_0$. Furthermore, since $\sim_1^A$ implies $\sim_1^V$, the following holds. If $\overline{v} \subseteq V_1$ with $\overline{v} \subseteq [u]_{\sim_1}$ for some $u \in V$, then for all $a \in \mathrm{act}(\overline{v})$ there is some $v \in V$ such that $Post_a(\overline{v}) \subseteq [v]_{\sim_1}$. Therefore we have $|\overline{v}\overline{E}_a| = 1$ for all $\overline{v} \in \overline{V}_1$ and all $a \in \mathrm{act}(\overline{v})$. So the game $\overline{G}_{v_0}$ is deterministic and each position of player 0 has exactly one successor position. Now we describe a nondeterministic procedure for solving the strategy problem for $\overline{G}_{v_0}$.

The procedure starts from $\{v_0\}$ and given some position $\overline{v} \in \overline{V}$ it works as follows. First, it nondeterministicly chooses whether to start circle-checking. If it does not, then it just nondeterministicly guesses some successor position of $\overline{v}$. (Notice that if $\overline{v} \in \overline{V}_0$ then this choice is actually deterministic.) If the procedure chooses to start circle-checking, then it defines $c := \mathrm{col}(\overline{v})$, $x := \overline{v}$ and guesses some successor position of $\overline{v}$. Now given c, x and $\overline{v}$, the procedure works as follows. If $\overline{v} = x$, then the procedure stops. If c is odd, then the procedure accepts, else it rejects. Now let $\overline{v} \neq x$. If $\mathrm{col}(\overline{v}) < c$, then the procedure defines $c := \mathrm{col}(\overline{v})$ and in any case it guesses some successor position of $\overline{v}$.

Obviously, the procedure accepts, if and only if there is a position $\overline{v}$ which is reachable from $\{v_0\}$ and which is furthermore reachable from $\overline{v}$ via some nonempty path, such that the least color on this path is odd. Thus, the procedure accepts if and only if player 1 has a winning strategy for $\overline{G}_{v_0}$ from $\{v_0\}$ which is equivalent to the existence of a winning strategy for player 1 for $\mathcal{G}$ from v_0. Clearly the strategy uses only polynomial space. Thus, using the Theorem of Savitch, the proof is finished. □

An immediate consequence of this result is that the strategy problem for blindfold Reif-games is in PSPACE. Now we want to show that the problem is PSPACE-complete. (A sketch of the original proof by Reif can be found in [Rei79].) For this, we need the following result.

Proposition 3.19. *Let $G = (V, E)$ be a Reif-game such that player 0 does never modify the common state. Then we can construct a blindfold Reif-game $G' = (V', E')$ with $V_1 \subseteq V_1'$ and $V_0' \subseteq V_0$ such that for each position $v_0 \in V \cap V'$ player 1 has a winning strategy from v_0 for G' if and only player 1 has a winning strategy from v_0 for G if private moves are hidden. If G is finite this construction can be done in time polynomial in the size of G.*

Proof. We define G' as follows. Let $E_{pr,0} := \{(v, w) \in E \mid\ v \in V_0,\ v \sim_1 w\ \}$ be the set of private moves of player 0 and let $V_{pr,0} := \{v \in V_0 \mid\ (w, v) \in E_{pr,0}$ for all $(w, v) \in E\ \}$ be the set of all positions v of player 0 such that all moves to v are private for player 0.

Now let $V' := (V \setminus V_{pr,0}) \uplus \{(1, *, *, *)\}$ and $E' := (E \setminus E_{pr,0}) \cup F$ where F contains the following edges. For positions $v \in V_0 \setminus V_{pr,0}$ and $w \in V_1$ such that there is a path $v = w_0 \to w_1 \to \ldots \to w_{n-1} \to w_n = w$ in G with $(w_i, w_{i+1}) \in E_{pr,0}$ for $i = 0, \ldots, n-2$ we let $(v, w) \in F$. For all $v \in V_0 \setminus V_{pr,0}$ such that there is an infinite path $v = w_0 \to w_1 \to \ldots$ with $(w_i, w_{i+1}) \in E_{pr,0}$ for all $i < \omega$ we let $(v, (1, *, *, *)) \in F$.

Since player 0 does not change the common state in the game G he does not change the common state in G' as well. Furthermore if $v \in V_0'$ and $(v, w) \in E'$ then $w \in V_1'$. So G' is a blindfold Reif-game.

The correctness of the construction can be shown with similar arguments as in the proof of proposition 3.14. ☐

Theorem 3.17. *The strategy problem for blindfold Reif-games is* PSPACE- *complete.*

Proof. Let $L \in PSPACE$, that means, there is a deterministic Turing-machine M with only one single tape and space bound n^k for some $k > 1$. Now we proceed exactly as in the proof of Theorem 3.15. We observe that since M is deterministic, for each input x of M the game $G(A, x)$ is a Reif-game where player $\forall$ does never modify the common part $c(C)$ of a position of $G(A, x)$. Now according to Proposition 3.19 we can construct a blindfold Reif-game $G' = (V', E')$ with $C_0(A, x) \in V'$ in time polynomial in $|G(A, x)|$ (and thus in time polynomial in $|x|$) such that player $\exists$ has a winning strategy for G' from initial position $C_0(A, x)$ if and only if player $\exists$ has a winning strategy for $G(A, x)$ from initial position $C_0(A, x)$ if private moves are hidden. Since M accepts input x if and only if player $\exists$ has a winning strategy for $G(A, x)$ from initial position $C_0(A, x)$ if private moves are hidden our proof is complete. ☐

3.6 Alternating Tree Automata

The idea of solving the strategy problem for games with partial information using alternating tree automata is taken from [KV99] where it has been used to solve the so called synthesis problem with incomplete information. There, no game graph is given explicitly but there are only actions, each of which can be chosen at each round of the game. The players move in strict alternation and the winning condition is given by a formula from CTL or CTL*. Of course this problem is a special case of the strategy problem for ω-regular games with partial information.

We use the idea to solve this problem in its full generality and we consider the complexity of this solution and the consequences concerning finite memory strategies in ω-regular games with partial information. Furthermore we shall see that, conversely, we can also use games with partial information to solve the nonemptiness problem for alternating tree automata. First, we introduce trees, tree automata and the widening operator along similar lines as in [KV99].

Let X be a set. An X-tree is a prefix closed set $T \subseteq X^*$. The tree T is called complete, if $T = X^*$. For a set Σ, a Σ-labelled X-tree is a function $t : T \to \Sigma$ for some X-tree T. The tree t is called complete if T is complete. Unless explicitly mentioned otherwise, we consider only complete trees here. By X_Σ we denote the set of all (complete) Σ-labelled X-trees.

For a set Y, the operator wide_Y maps Σ-labelled X-trees to Σ-labelled $(X{\times}Y)$-trees. To make the notation more simple we usually consider $(X{\times}Y)^*$ as a subset of $X^*{\times}Y^*$, where $(x_1, y_1) \dots (x_n, y_n)$ is identified with $(x_1 \dots x_n, y_1 \dots y_n)$. Now for a Σ-labelled X-tree $t : X^* \to \Sigma$, the tree $\mathrm{wide}_Y(t) : (X{\times}Y)^* \to \Sigma$ is defined by $\mathrm{wide}_Y(t)(\varepsilon) = t(\varepsilon)$ and $\mathrm{wide}_Y(t)((x, y)) = t(x)$ for all $(x, y) \in (X{\times}Y)^*$.

Now let X and Σ be finite. An alternating tree automaton over Σ-labelled X-trees is given by a tuple $\mathcal{A} = (\Sigma, Q, \delta, q_0, \mathrm{acc})$ with a finite set Q of states, a transition function $\delta : Q{\times}\Sigma \to B^+(X{\times}Q)$, where $B^+(X{\times}Q)$ is the set of all positive Boolean formulas over variables from $X \times Q$, and an acceptance component $\mathrm{acc} \subseteq Q^\omega$.

A run of $\mathcal{A}$ on a Σ-labelled X-tree $t : X^* \to \Sigma$ is a not necessarily complete Σ_r-labelled ω-tree $t_r : T_r \to \Sigma_r$ where $\Sigma_r = X^* \times Q$, such that the following conditions hold.

(1) $\varepsilon \in T_r$ and $t_r(\varepsilon) = (\varepsilon, q_0)$.

(2) If $y \in T_r$ with $t_r(y) = (x, q)$ and $\delta(q, t(x)) = \varphi$, then there is a (possibly empty) set $S = \{(x_0, q_0), \dots, (x_{n-1}, q_{n-1})\} \subseteq X \times Q$ such that the following conditions hold.

  (2.1) $[\![\varphi]\!]^{\mathcal{I}} = 1$, where for $V \in X \times Q$ we have $\mathcal{I}(V) = 1$ if and only if $V \in S$.

  (2.2) For all $0 \leq i < n$ we have $y \cdot i \in T_r$ and $t_r(y \cdot i) = (x \cdot x_i, q_i)$.

The run t_r is called accepting, if for each infinite path $n_1(n_1n_2)(n_1n_2n_3)\dots$ through T, the sequence $\gamma \in Q^\omega$ with $\gamma(i) = \mathrm{pr}_2(t_r(n_1n_2 \dots n_i))$ for $i < \omega$ belongs to acc. The automaton $\mathcal{A}$ accepts a Σ-labelled X-tree t, if there is an accepting run of $\mathcal{A}$ on t. The language of $\mathcal{A}$ is $L(\mathcal{A}) = \{t \in X_\Sigma \mid \mathcal{A} \text{ accepts } t\}$.

Now w.l.o.g. let $X = \{1, \dots, k\}$ for some $k < \omega$. The automaton $\mathcal{A}$ is called nondeterministic, if for each $(q, a) \in Q \times \Sigma$, the disjunctive normal form of $\delta(q, a)$ has the form $(q_1^1, 1) \wedge \dots \wedge (q_k^1, k) \vee \dots \vee (q_1^n, 1) \wedge \dots \wedge (q_k^n, k)$ for some $n < \omega$. The automaton is called deterministic, if for each such disjunctive normalform we have $n = 1$. The automaton is called universal, if for each $(q, a) \in Q \times \Sigma$, the disjunctive normalform of $\delta(q,a)$ has the form $(q^1_1,1) \wedge \ldots \wedge (q^1_{l_1},1) \wedge \ldots \wedge (q^k_1,k) \wedge \ldots \wedge (q^k_{l_k},k)$ where $l_i < \omega$ for $i \in \{1,\ldots,k\}$.

We collect some results about tree automata, in order to get a better understanding of the results that we derive in the next sections. Muller tree automata, parity tree automata and (co-) Büchi automata are defined in the same way as the corresponding ω-automata.

Theorem 3.18. *(1) Nondeterministic Muller tree automata and nondeterministic parity tree automata are equally expressive.*

(2) Alternating Muller tree automata are equally expressive to nondeterministic Muller tree automata. The same holds for parity tree automata and for Büchi tree automata.

(3) universal Muller tree automata are equally expressive to deterministic Muller tree automata. The same holds for parity tree automata and for Büchi tree automata.

(4) Deterministic Muller tree automata are less expressive than nondeterministic Muller tree automata. The same holds for parity tree automata and for Büchi tree automata.

(5) The class of nondeterministicly parity recognizable tree languages is closed under complement. This does not hold for the class of Büchi-recognizable tree languages.

Proof. (1) : [GTW02]. (2), (3) : [MS95]. (4) We define $t_1, t_2 \in \{0,1\}_{\{a,b\}}$ as follows. $t_1(u) = a$, if $u = 0u'$ for some $u' \in \{0,1\}^*$ and $t_1(u) = b$, else and $t_2(u) = a$, if $u = 1u'$ for some $u' \in \{0,1\}^*$ and $t_2(u) = b$, else. Now it is easy to see that the language $L := \{t_1, t_2\} \subseteq \{0,1\}_{\{a,b\}}$ is not deterministicly Muller recognizable but nondeterministicly Büchi-recognizable. (5) First part : [GTW02]. Second part : $L := \{t \in \{0,1\}_{\{a,b\}} \mid t \text{ contains a path with infinitely many } b\,\}$ is Büchi-recognizable (easy), while the complement of L isn't ([GTW02]). ☐

3.6.1 From Games to Automata

Let $\mathcal{G} = (G, (\mathrm{vis}^V_1), (\mathrm{vis}^A_1))$, $G = (V, V_0, (f_a)_{a\in A}, W_0)$ be a finite game with partial information and let $v_0 \in V$ be some initial position. As a technical simplification, we assume w.l.o.g. that $v_0 \in V_1$.

A strategy $f : \{\pi \in P_{\mathrm{fin}}(v_0) \mid \mathrm{last}(\pi) \in V_1\} \to A$ for player 1 for G from v_0 can be represented as an A-labelled (AV)-tree $t_f : (AV)^* \to A$ by $t_f(\pi) = f(v_0\pi)$, if $v_0\pi \in \mathrm{dom}(f)$. For each $\pi \in (AV)^*$ with $v_0\pi \notin \mathrm{dom}(f)$, let $t_f(\pi)$ be an arbitrary action from A. Conversely we obtain a unique strategy for player 1 for G from v_0, from each A-labelled (AV)-tree $t : (AV)^* \to A$ such that for all $\pi \in (AV)^*$ with $v_0\pi \in P_{\mathrm{fin}}(v_0)$ and $\mathrm{last}(v_0\pi) \in V_1$ we have $t(\pi) \in \mathrm{act}(\mathrm{last}(v_0\pi))$.

However, this representation is not appropriate for our concerns. Instead, we represent a strategy f for player 1 for G from v_0 as an A-labelled $(\mathrm{VIS}^A_1\,\mathrm{VIS}^V_1 \times AV)$-tree $t_f : (\mathrm{VIS}^A_1\,\mathrm{VIS}^V_1 \times AV)^* \to A$ in the following way. First we define $t_f(\varepsilon) = f(v_0)$. Now for a sequence $\pi = (x_1y_1, a_1v_1) \ldots (x_ny_n, a_nv_n) \in (\mathrm{VIS}^A_1\,\mathrm{VIS}^V_1 \times AV)^*$ we define $t_f(\pi) = f(v_0a_1v_1 \ldots a_nv_n)$, if $v_0a_1v_1 \ldots a_nv_n \in \mathrm{dom}(f)$ and $\mathrm{vis}_1(a_iv_i) = x_iy_i$

for all $i \in \{1, \ldots, n\}$. If $v_0 a_1 v_1 \ldots a_n v_n \notin \mathrm{dom}(f)$ or $\mathrm{vis}_1(a_i v_i) \neq x_i y_i$ for some $i \in \{1, \ldots, n\}$, let $t_f(\pi)$ be an arbitrary action from A. Conversely, let t be an A-labelled $(\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV)$-tree such that the following holds. For each sequence $\pi = (x_1 y_1, a_1 v_1) \ldots (x_n y_n, a_n v_n) \in (\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV)^*$ with $v_0 a_1 v_1 \ldots a_n v_n \in P_{\mathrm{fin}}(v_0)$, $v_n \in V_1$ and $\mathrm{vis}_1(a_i v_i) = x_i y_i$ for $i = 1, \ldots, n$ we have $t(\pi) \in \mathrm{act}(v_n)$. Then from t we obtain a strategy for player 1 for G from v_0 in the obvious way.

Now we construct a deterministic tree automaton $\mathcal{A} = (A, Q, \delta, v_0, \mathrm{acc})$ which accepts an A-labelled $(\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV)$-tree t if and only if f_t is a winning strategy for player 1 for G from v_0. Notice that we are asking for *full information* winning strategies here. We define the components of the automaton as follows.

- $Q = \{v_0\} \uplus \mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV \uplus \{q_+, q_-\}$.
- $\delta(v_0, a, (xy, bv)) =$
  - (xy, bv), if $a = b \in \mathrm{act}(w)$ and $f_b(w) = v$ and $\mathrm{vis}_1(bv) = xy$.
  - q_-, if $a \notin \mathrm{act}(v_0)$.
  - q_+, otherwise.
- $\delta((pq, cw), a, (xy, bv)) =$
  - (xy, bv), if $w \in V_0$ and $b \in \mathrm{act}(w)$ and $f_b(w) = v$ and $\mathrm{vis}_1(bv) = xy$.
  - (xy, bv), if $w \in V_1$ and $a = b \in \mathrm{act}(w)$ and $f_b(w) = v$ and $\mathrm{vis}_1(bv) = xy$.
  - q_-, if $w \in V_1$ and $a \notin \mathrm{act}(w)$.
  - q_+, otherwise.
- $\delta(q_+, a, (xy, bv)) = q_+$.
- $\delta(q_-, a, (xy, bv)) = q_-$.
- $\mathrm{acc} = Q^* q_+^\omega \cup \{(x_i y_i, a_i v_i)_{i<\omega} \mid v_0 (a_i v_i)_{i<\omega} \in W_1\}$.

So for an A-labelled $(\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV)$-tree t, the automaton checks, whether the function f_t as we have defined it above, is indeed a strategy for player 1. If there is a node in t, such that at the last position of the corresponding play prefix, player 1 has to move, but the labelling of the node is not consistent with the actions in the game, then the automaton rejects by labelling all the paths in t which start in this node with q_-. Furthermore, the automaton labels all the paths in t that correspond to plays in G from v_0 which are compatible with f_t, with the nodes of the tree in order to check, whether each such play in won by player 1. Each path in t which does not correspond to such a play is labelled with q_+ from the point at which it does not correspond to such a play anymore, so it is not examined any further.

Therefore, $\mathcal{A}$ accepts an A-labelled $(\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V \times AV)$-tree t if and only if f_t is a winning strategy for player 1 for G from v_0. In particular, $L(\mathcal{A}) \neq \emptyset$ if and only if player 1 has a winning strategy for G from v_0. Clearly we can construct the automaton $\mathcal{A}$ in time polynomial in the size of $\mathcal{G}$. Furthermore, if G is a Muller-game, then $\mathcal{A}$ is a Muller tree automaton and in the same way, the construction transforms parity conditions into parity conditions and (co-) Büchi-conditions into (co-) Büchi-conditions.

Theorem 3.19. ([KV99]) *Let X, Y and Σ be finite sets. Given an alternating tree automaton $\mathcal{B}$ over Σ-labelled $(X \times Y)$-trees, one can effectively construct an alternating tree automaton $\mathcal{B}'$ over Σ-labelled X-trees, such that the following holds.*

(1) $\mathcal{B}'$ accepts a tree t if and only if $\mathcal{B}$ accepts $\mathrm{wide}_Y(t)$.
(2) $\mathcal{B}'$ and $\mathcal{B}$ have the same acceptance component.

The construction can be done in time linear in the size of $\mathcal{B}$.

Remark. Even if the automaton $\mathcal{B}$ is deterministic, the automaton $\mathcal{B}'$ which is constructed in the proof of Theorem 3.19 (cf. [KV99]) may be seriously alternating. However, if $\mathcal{B}$ is *universal*, then so is $\mathcal{B}'$.

Now consider the automaton $\mathcal{A}$ that we have constructed above and let $\mathcal{A}'$ be the corresponding automaton according to Theorem 3.19. (Where $X = \mathrm{VIS}_1^A\,\mathrm{VIS}_1^V$, $Y = AV$ and $\Sigma = A$.) The following result yields the desired solution of the strategy problem for games with partial information.

Proposition 3.20. *$L(\mathcal{A}') \neq \emptyset$ if and only if $v_0 \in \mathrm{Win}_1^{\mathcal{G}}$.*

Proof. First let $L(\mathcal{A}') \neq \emptyset$ and let t be an A-labelled $\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V$-tree such that $t \in L(\mathcal{A}')$. Then $t' := \mathrm{wide}_{AV}(t) \in L(\mathcal{A})$, that means, $f_{t'}$ is a winning strategy for player 1 for G from v_0. Now let $\pi = v_0 a_1 v_1 \dots a_n v_n$, $\pi' = v_0 b_1 w_1 \dots b_n w_n \in \mathrm{dom}(f_{t'})$ with $\pi \sim_1^* \pi'$ and consider $\sigma := (\mathrm{vis}_1(a_1 v_1), a_1 v_1) \dots (\mathrm{vis}_1(a_n v_n), a_n v_n)$ and $\sigma' := (\mathrm{vis}_1(b_1 w_1), b_1 w_1) \dots (\mathrm{vis}_1(b_n w_n), b_n w_n)$. Then $\tau := \mathrm{vis}_1(a_1 v_1) \dots \mathrm{vis}_1(a_n v_n) = \mathrm{vis}_1(b_1 w_1) \dots \mathrm{vis}_1(b_n w_n) =: \tau'$ and so $f_{t'}(\pi) = t'(\sigma) = t(\tau) = t(\tau') = t'(\sigma') = f_{t'}(\pi')$. Thus, $f_{t'}$ is a winning strategy for player 1 for $\mathcal{G}$ from v_0.

Now let conversely f be a winning strategy for player 1 for $\mathcal{G}$ from v_0. We define the A-labelled $\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V$-tree t as follows. For each sequence $\sigma = x_1 y_1 \dots x_n y_n \in (\mathrm{VIS}_1^A\,\mathrm{VIS}_1^V)^*$ such that there is some $\pi \in \mathrm{dom}(f)$ with $\mathrm{vis}_1^*(\pi) = \mathrm{vis}_1(v_0)\sigma$ we define $t(\sigma) = f(\pi)$ for any such π. If there is no such π, let $t(\sigma)$ be an arbitrary action from A. Now let $t' := \mathrm{wide}_{AV}(t)$. It is easy to see that $f_{t'} = f$, so $\mathcal{A}$ accepts t' and thus, $\mathcal{A}'$ accepts t. This yields $L(\mathcal{A}') \neq \emptyset$. ☐

The following result can now be formulated as a corollary of Theorem 3.15.

Corollary 3.4. *The nonemptiness problem for universal Büchi tree automata is* EXPTIME-*hard.*

3.6.2 Complexity and Finite Memory

So far we have seen that we can solve the strategy problem for finite games with partial information using alternating tree automata, if the nonemptiness problem for the corresponding class of alternating tree automata is decidable. This is the case for parity tree automata. Furthermore, if a parity tree automaton accepts some tree, then there is also a finite state automaton which generates such a tree. Now we shall have look at certain results concerning the complexity of testing nonemptiness of parity tree automata and of computing such a finite state tree.

First notice that we do not need the full expressive power of alternating tree automata, since for each game $\mathcal{G}$ with partial information and each initial position v_0 there is a *universal* tree automaton, such that the language of this automaton is nonempty, if and only if player 1 has a winning strategy for $\mathcal{G}$ from v_0. (From Theorem 3.18 it follows that in general, universal tree automata are strictly less expressive than nondeterministic tree automata.)

In [KV05], the nonemptiness problem for alternating parity tree automata is solved without using the classical constructions for removing alternation from tree automata by Safra [Saf88] and Muller-Schupp [MS95]. There, an alternating parity tree automaton is transformed into a universal co-Büchi tree automaton, such that the language of the latter is nonempty if and only if the language of the former is nonempty. Then, from the resulting automaton a nondeterministic Büchi tree automaton is constructed, such that again, nonemptiness of the language of the automaton is preserved. (From Theorem 3.18 it follows that in general, for a given alternating parity tree automaton, we cannot construct an *equivalent* nondeterministic Büchi tree automaton.)

So if we want to solve the strategy problem for co-Büchi-games with partial information using this construction, then we just have to carry out the construction step from universal co-Büchi tree automata to nondeterministic Büchi tree automata. (The nonemptiness problem for the latter can be solved in quadratic time.) Moreover, in [KPV06], the construction has been extended to generalized co-Büchi tree automata, where we do not have a single set F, but sets $F_1, \ldots, F_k$, at least one of which has to be seen only finitely often in an accepting run.

This might give better algorithms for solving the strategy problem for (generalized) co-Büchi-games with partial information. Of course the method is also of great general interest, but we do not consider this here in more detail. Instead we shall have a look at the result of Muller-Schupp and the so called Regularity Theorem.

First we have to introduce the Streett acceptance condition. A Streett tree automaton has the form $\mathcal{A} = (\Sigma, Q, \delta, q_0, \{(L_k, U_k) \,|\, k = 1, \ldots, r\})$ with $L_k, U_k \subseteq Q$ for $k = 1, \ldots, r$. The acceptance component acc is defined as follows. For $\pi = q_0q_1q_2 \ldots \in Q^\omega$, $\pi \in$ acc if and only if for all $k \in \{1, \ldots, r\}$ we have $U_k \cap \inf(\pi) = \emptyset$ or $L_k \cap \inf(\pi) \neq \emptyset$. The Streett condition is the dual of the Rabin condition. If the sets $L_1, \ldots, L_r$ form a chain $L_1 \subseteq \ldots \subseteq L_r$ then there is a finite coloring of Q with at most $2r$ colors, such that the corresponding parity condition coincides with acc. Vice versa, for each parity condition with at most $2r$ colors, there are sets $L_1 \subseteq \ldots \subseteq L_r \subseteq Q$ and $U_1, \ldots, U_r \subseteq Q$ such that the corresponding Streett condition coincides with the parity condition.

Theorem 3.20. (Muller-Schupp, [MS95]) *Let $\mathcal{A}$ be an alternating parity tree automaton with n states and k colors. Then one can construct a nondeterministic Streett tree automaton $\mathcal{A}'$ with at most $2^{O(kn\log(n))}$ states and at most $O(kn\log(n))$ pairs (L, U) such that $L(\mathcal{A}') = L(\mathcal{A})$. If $\mathcal{A}$ is universal, then $\mathcal{A}'$ is deterministic.*

Remark. It can be seen that this construction is effective and can be done in time exponential in the size of the original automaton $\mathcal{A}$.

Theorem 3.21. ([BLV96]) *Let $\mathcal{A}$ be a Streett tree automaton with n states and r pairs (L, U). Then one can construct a parity tree automaton $\mathcal{A}'$ with at most $n \cdot r!$ states and at most $2r$ colors such that $L(\mathcal{A}') = L(\mathcal{A})$. The construction can be done in time $O(n^2 r!)$. If $\mathcal{A}$ is nondeterministic (deterministic, respectively) then $\mathcal{A}'$ is nondeterministic (deterministic, respectively) as well.*

Corollary 3.5. *Let $\mathcal{A}$ be a universal parity tree automaton with n states and k colors and let $C := k \cdot n \cdot \log(n)$. Then one can construct a deterministic parity tree automaton $\mathcal{A}'$ with $2^{O(C \log(C))}$ states and $O(C)$ colors such that $L(\mathcal{A}') = L(\mathcal{A})$. The construction can be done in time exponential in the size of $\mathcal{A}$.*

This yields, for each finite parity game with partial information, a deterministic parity tree automaton such that the following holds. The number of states of the automaton is exponential in the size of the game, the number of colors of the automaton is polynomial in the size of the game and player 1 has a winning strategy for the game if and only if the language of the automaton is nonempty. Now we solve the nonemptiness problem for the automaton by constructing a parity game, such that player 1 has a winning strategy for the game if and only if the language of the automaton is nonempty. Now, of course, we construct a game with full information. This is done as follows.

Let $\mathcal{A} = (\Sigma, Q, \Delta, q_0, \mathrm{col})$ be a nondeterministic parity tree automaton where $\Delta \subseteq Q \times \Sigma \times Q^{\Sigma}$ and $\Sigma = \{1, \ldots, m\}$. (Notice that each nondeterministic parity tree automaton can be represented in this form.) We define the parity game $G(\mathcal{A}) = (V, V_0, E, \mathrm{col})$ (with $E \subseteq V \times V$) by $V = \Delta \uplus Q$, $V_0 = \Delta$, $E = \{(q, (q, a, q_1, \ldots, q_m)) \mid (q, a, q_1, \ldots, q_m) \in \Delta\ \} \cup \{((q, a, q_1, \ldots, q_m), q_i) \mid (q, a, q_1, \ldots, q_m) \in \Delta, i \in 1, \ldots, m\}$ and $\mathrm{col}((q, a, q_1, \ldots, q_m)) = \mathrm{col}(q)$ for $(q, a, q_1, \ldots, q_m) \in \Delta$. Then $L(\mathcal{A}) \neq \emptyset$ if and only if player 1 has a winning strategy for $G(\mathcal{A})$ from q_0. Now if player 1 has such a winning strategy, then he also has a positional winning strategy, since $G(\mathcal{A})$ is a parity game. Furthermore, from a positional winning strategy for player 1 for $G(\mathcal{A})$ from q_0 we can construct a finite automaton with output, which generates some tree that is accepted by $\mathcal{A}$. Such trees are called regular. The result has first been proved by Rabin in [Rab72].

Theorem 3.22. *(Regularity Theorem) Let X and Σ be finite sets and let $\mathcal{A} = (\Sigma,$ $Q,$ $\Delta,$ $q_0,$ $\mathrm{col})$ be a nondeterministic parity tree automaton with $L(\mathcal{A}) \neq \emptyset$. Then there is a finite deterministic word automaton $\mathcal{B} = (X, Q_{\mathcal{B}}, \delta_{\mathcal{B}}, q_{0\mathcal{B}}, \tau_{\mathcal{B}})$ with output function $\tau_{\mathcal{B}} : Q_{\mathcal{B}} \to \Sigma$ and state set $Q_{\mathcal{B}} = \Delta$, such that the tree $t_{\mathcal{B}} : X^* \to \Sigma$ with $t_{\mathcal{B}}(w) = \tau_{\mathcal{B}}(\delta^*_{\mathcal{B}}(w))$ is in $L(\mathcal{A})$. The automaton $\mathcal{B}$ can be constructed from $\mathcal{A}$ in time $|\Delta|^{O(k)}$ where $k = |\operatorname{im}(\mathrm{col})|$.*

Proof. Let $\Sigma = \{1, \ldots, m\}$ and let $f : Q \to \Delta$ be a positional winning strategy for player 1 for $G(\mathcal{A})$ from q_0. We define $q_{0\mathcal{B}} = f(q_0)$ and $\delta_{\mathcal{B}}((q, a, q_1, \ldots, q_m), k) = f(q_k)$ for $(q, a, q_1, \ldots, q_m) \in \Delta$ and $k \in \{1, \ldots, m\}$. Moreover, we define $\tau(q, a, q_1, \ldots, q_m)$ $= a$ for $(q, a, q_1, \ldots, q_m) \in \Delta$. By construction, $\mathcal{A}$ accepts $t_{\mathcal{B}}$ and since we can construct f from $\mathcal{A}$ in time $|\Delta|^{O(k)}$, the theorem is proved. □

So, starting from a finite parity game $\mathcal{G}$ with partial information and some initial position $v_0 \in V$, we can decide in time exponential in $|V|$, whether player 1 has a winning strategy for $\mathcal{G}$ from v_0. If so, we can construct a finite deterministic automaton $\mathcal{B}$ (with state set Q, such that $|Q|$ is exponential in $|V|$) which generates a tree $t_{\mathcal{B}}$ that is accepted by the automaton $\mathcal{A}'$ as in Theorem 3.19. Hence, the tree $\text{wide}_{AV}(t_{\mathcal{B}})$ is accepted by $\mathcal{A}$ and therefore, $\mathcal{B}$ yields a finite memory winning strategy for player 1 for $\mathcal{G}$ from v_0.

Corollary 3.6. *The strategy problem for parity games with partial information and n states can be solved in time exponential in n. Furthermore, if player* 1 *has a winning strategy from some initial position, then he has a finite memory winning strategy from this position such that the size of the memory structure is exponential in n and the memory structure can be constructed in time exponential in n.*

3.6.3 From Automata to Games

Now we shall see how we can solve the nonemptiness problem for alternating automata using games with partial information. This might seem to be somehow the converse of the automata theoretic solution of games with partial information. But in general it is not. Notice that we have already observed that for solving games with partial information, *universal* tree automata suffice. And we will see that for such tree automata, two-player games with partial information are appropriate for solving the nonemptiness problem. But for tree automata which are alternating in an unrestricted way, those games do not suffice in general.

Consider an alternating tree-automaton $\mathcal{A}$. We have $L(\mathcal{A}) \neq \emptyset$ if and only if there *exists* an input tree t for $\mathcal{A}$ such that there *exists* a run of $\mathcal{A}$ on t such that *for each* (infinite) path through this run, this path is accepting. So consider the following game played between an existential player and a universal player. The existential player guesses an input tree t for $\mathcal{A}$ and at the same time guesses an accepting run of $\mathcal{A}$ on t. The universal player chooses the branches of the run which he wants to be verified. Now if there is some tree which is accepted by $\mathcal{A}$, then clearly the existential player has a winning strategy for the game. Vice versa, if the existential player has a winning strategy for the game, then there is an input tree which is accepted by $\mathcal{A}$, if $\mathcal{A}$ is *nondeterministic*. The game that we have described is exactly the nonemptiness game for nondeterministic tree automata. If $\mathcal{A}$ is seriously alternating, then the existential player may have a winning strategy for the game, even if $L(\mathcal{A}) = \emptyset$.

The reason for this is simple. The existential player may choose the labelling of the input tree which he guesses in dependence of the previous moves of the universal player, and this is of course not correct. The information that we have to provide to the existential player for guessing the input tree are the branches of the *input tree* that the universal player chooses. But the existential player must not know the branches of the run that the universal player chooses. On the other hand, for guessing the run of the automaton on the input tree, we have to provide full information to the existential player. Since a player cannot have full information and partial information at the same time, we have to split the existential part of the game into two players, one of which guessing the input tree, having partial information and the other guessing the run of the automaton, having full information. Then, the language of the automaton is nonempty, if and only if the existential players can *cooperate* to win. So we have a three player win-loss game with partial information which is not a zero-sum game. The winning conditions for the existential players coincide and the winning condition for the universal player is the complement of this set. We call the existential players tree (T), guessing the input tree and automaton (A), guessing the run.

Let $\mathcal{A} = (\Sigma, Q, \delta, q_0, \mathrm{acc})$ be an alternating tree automaton over Σ-labelled X-trees for some finite set $X = \{1, \dots, k\}$ and some finite set Σ. We define the three-player game $G(\mathcal{A}) = (V, V_T, V_A, (f_a), W_\forall)$ as follows.

- $V = V_T \uplus V_A \uplus V_\forall$.
  - $V_T = \{(q_0, 0)\} \cup Q \times X$.
  - $V_A = \{(v, a) \mid v \in V_T, a \in \mathrm{act}(v)\}$.
  - $V_\forall = \{(v, a) \mid v \in V_A, a \in \mathrm{act}(v)\}$.
- For $v \in V_T$, $\mathrm{act}(v) = \Sigma$ and $f_a(v) = (v, a)$ for all $a \in \mathrm{act}(v)$.
- For $v = ((q, r), \sigma) \in V_A$, $\mathrm{act}(v)$ is the set of all tuples $((q_1^1, 1), \dots, (q_{l_1}^1, 1), \dots, (q_1^k, k), \dots, (q_{l_k}^k, k))$ such that $(q_1^1, 1) \wedge \dots \wedge (q_{l_1}^1, 1) \wedge \dots \wedge (q_1^k, k) \wedge \dots \wedge (q_{l_k}^k, k)$ is a conjunct in the disjunctive normalform of $\delta(q, \sigma)$.
- For $v \in V_A$ and $a \in \mathrm{act}(v)$, $f_a(v) = (v, a)$.
- For $w = (v, a) \in V_\forall$ with $a = ((q_1^1, 1), \dots, (q_{l_k}^k, k))$, $\mathrm{act}(w) = \{(i, j) \mid 1 \le i \le k, 1 \le j \le l_i\}$ and $f_{(i,j)}(w) = (q_j^i, i)$ for all $(i, j) \in \mathrm{act}(w)$.
- For a play $\pi = v_0 a_0 v_1 a_1 v_2 \dots$ in $G(\mathcal{A})$ from initial position $v_0 = (q_0, 0)$ we define $\pi \in W_\forall$ if and only if $q_0 q_1 q_2 \dots \notin \mathrm{acc}$, where $v_{3i} = (q_i, j)$ for all $i < \omega$.

The information of player T in the game is defined as follows.

- $\mathrm{vis}_T^V(q, i) = i$ for all $(q, i) \in V_T$.
- $\mathrm{vis}_T^V(v, a) = \mathrm{vis}_T^V(v)$ for all $(v, a) \in V_A$.
- $\mathrm{vis}_T^V(v, a) = \mathrm{vis}_T^V(v)$ for all $(v, a) \in V_\forall$.
- $\mathrm{vis}_T^A(a) = a$ for all actions a of player T.
- $\mathrm{vis}_T^A(a) = *$ for some symbol $*$, for all actions a of player A and player $\forall$.

So player T sees only the branches of the input tree that have been chosen by the universal player. Of course he can distinguish all his own actions, but he cannot distinguish any other actions. Player A and player $\forall$ have full information. We call the corresponding game with partial information $\mathcal{G}(\mathcal{A})$. Clearly we can construct $\mathcal{G}(\mathcal{A})$ in time polynomial in the size of $\mathcal{A}$.

Notice that there may be finite plays in this game. Such a play ends in a position of player A or of player $\forall$. A finite play which ends in a position of player A is won by the universal player (in this case the disjunct from which A has to choose a conjunct is empty, so the formula at hand is *false*), a finite play which ends in a position of player $\forall$ is lost by the universal player (in this case the conjunct from which $\forall$ has to choose a literal is empty, so the formula at hand is *true*).

Theorem 3.23. *Let X and Σ be finite sets and let $\mathcal{A} = (\Sigma, Q, \delta, q_0, \mathrm{acc})$ be an alternating tree automaton over Σ-labelled X-trees. Then the following statements are equivalent.*

(1) $L(\mathcal{A}) \neq \emptyset$.
(2) There are strategies f for player T for $\mathcal{G}(\mathcal{A})$ from $(q_0, 0)$ and g for player A for $\mathcal{G}(\mathcal{A})$ from $(q_0, 0)$ such that each play in $\mathcal{G}(\mathcal{A})$ from $(q_0, 0)$ that is compatible with f and g is lost by player $\forall$.

Proof. First let (1) hold, that means, there is a tree $t : X^* \to \Sigma$ such that $t \in L(\mathcal{A})$. The latter means there is a run $t_r : T_r \to \Sigma_r$ of $\mathcal{A}$ on t where $T_r \subseteq \omega^*$ and $\Sigma_r = X^* \times Q$. We define the strategies f and g as follows.

First, let $\pi = v_0 a_0 v_1 a_1 \ldots a_{3n-1} v_{3n}$ be a finite prefix of a play in $\mathcal{G}(\mathcal{A})$ from $v_0 = (q_0, 0)$ for some $n < \omega$, that means, it is player T's turn at the last position of the prefix. Now consider the sequence $x := \mathrm{vis}_T^V(v_0)\,\mathrm{vis}_T^A(a_0) \ldots \mathrm{vis}_T^A(a_{3n-1})\,\mathrm{vis}_T^V(v_{3n})$ and let $i_1, \ldots, i_n \in \{1, \ldots, k\}$ and $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $x = 0\sigma_1 0{*}0{*}i_1\sigma_2 \ldots {*}i_n$. We define $f(\pi) := t(i_1 \ldots i_n)$.

Now we define the value of the strategy g for player A on all finite prefixes $\pi_n = v_0 a_0 v_1 a_1 \ldots a_{3n} v_{3n+1}$ of plays in $\mathcal{G}(\mathcal{A})$ from $v_0 = (q_0, 0)$ which are compatible with g and f, inductively over n, and with each such prefix we associate a sequence $j_1 \ldots j_n \in T_r$ such that the following holds. First, $t_r(j_1 \ldots j_n) = (i_1 \ldots i_n, q)$ where $i_1, \ldots, i_n$ are the elements that we have chosen in the definition of f and $v_{3n} = (q, i_n)$, with $i_0 = 0$. Second, if $\delta(q, t(i_1 \ldots i_n)) = \varphi$ and $S = \{(x_0, q_0), \ldots, (x_{n-1}, q_{n-1})\} \subseteq X \times Q$ is the (possibly empty) set from the definition of a run, then player A chooses a conjunct φ' from the disjunctive normal form of φ which is satisfied by S. And finally, if $\pi_n \preceq \pi_{n+1}$ and $j_1 \ldots j_n$ is the sequence associated with π_n, then the sequence associated with π_{n+1} is $j_1 \ldots j_n j_{n+1}$ for some $j_{n+1} < \omega$.

First let $n = 0$. The sequence $j_1 \ldots j_0$ associated with $\pi_0 = v_0\ a_0\ v_1$ is empty. Now let $\delta(q_0, t(\varepsilon)) = \delta(q_0, \sigma_1) = \varphi$. Then there is a set $S = \{(x_0, q_0), \ldots, (x_{m-1}, q_{m-1})\} \subseteq X \times Q$ such that $[\![\varphi]\!]^{\mathcal{I}} = 1$, where for $V \in X \times Q$ we have $\mathcal{I}(V) = 1$ if and only if $V \in S$. Since $[\![\varphi]\!]^{\mathcal{I}} = 1$, there is a conjunct φ' in the disjunctive normal form of φ, such that $[\![\varphi']\!]^{\mathcal{I}} = 1$ and we define $g(\pi_0) := \varphi'$.

Now let $n > 0$, let $v_{3n-1} = ((q, i_{n-1}), \sigma_n), \varphi')$ with $\varphi' = (q_1^1, 1) \wedge \ldots \wedge (q_{l_k}^k, k)$ and let $a_{3n-1} = (i, j) \in \{(i', j') \mid 1 \leq i' \leq k, 1 \leq j' \leq l_i\}$. Then we have $t_r(j_1 \ldots j_{n-1}) = (i_1 \ldots i_{n-1}, q)$ where $v_{3(n-1)} = (q, i_{n-1})$. Furthermore, let $\delta(q, t(i_1 \ldots i_{n-1})) = \varphi$ and let $S = \{(x_0, q_0), \ldots, (x_{m-1}, q_{m-1})\}$ be the set from the definition of a run. Since π_n is compatible with g, φ' is a conjunct in the disjunctive normal form of φ which is satisfied by S. So if $(x_r, q_r) \in S$ such that $x_r = i$ and $q_r = q_j^i$, then for $j_n = r$ we have $j_1 \ldots j_{n-1} j_n \in T_r$ and $t_r(j_1 \ldots j_{n-1} j_n) = (i_1 \ldots i_{n-1} i, q_j^i)$, where $(q_j^i, i_n) = v_{3n} = (q_j^i, i)$.

Now let $\delta(q_j^i, t(i_1 \ldots i_n)) = \delta(q_j^i, \sigma_{n+1}) = \varphi$, where the former equality is due to the fact that π_n is compatible with f. Since $t_r(j_1 \ldots j_n) = (i_1 \ldots i_n, q_j^i)$, there must be some set $S = \{(x_0, q_0), \ldots, (x_{m-1}, q_{m-1})\} \subseteq X \times Q$ such that $[\![\varphi]\!]^{\mathcal{I}} = 1$ and so, there is a conjunct φ' in the disjunctive normal form of φ with $[\![\varphi']\!]^{\mathcal{I}} = 1$. We define $g(\pi_n) := \varphi'$.

Now we show that each play in $\mathcal{G}(\mathcal{A})$ from $(q_0, 0)$ which is compatible with f and g is lost by player $\forall$. So let $\pi = v_0 a_0 v_1 a_1 \ldots$ be any such play. If the play is finite, then it has to be lost by player $\forall$, since for each finite prefix of a play in $\mathcal{G}(\mathcal{A})$ from $(q_0, 0)$ which is compatible with f and g, where it is player A's turn, we have found an action which player A can choose. Now let π be infinite and for $n < \omega$ consider the sequence $j_1 \ldots j_n$ associated with $\pi_n = v_0 a_0 v_1 a_1 \ldots a_{3n} v_{3n+1}$. Then $j_1 j_2 \ldots$ is an infinite path through T_r and thus, the sequence $\gamma \in Q^\omega$ which is defined via $\gamma(n) := \mathrm{pr}_2(t_r(j_1 \ldots j_n))$ for any $n < \omega$ belongs to acc. But now we have $t_r(j_1 \ldots j_n) = (i_1 \ldots i_n, q)$ where $v_{3n} = (q, i_n)$ for all $n < \omega$ and hence, $\pi \notin W_\forall$.

If conversely (2) holds, then a very similar kind of reasoning as before shows that (1) holds as well. ☐

The Universal Case. Now consider the case where $\mathcal{A}$ is *universal*. Then at each position of player A, only a single move is possible and thus, we can eliminate all positions of player A from the game. By doing so, we obtain a two-player game with partial information such that player $T = \exists$ has a winning strategy for the game from $(q_0, 0)$, if and only if $L(\mathcal{A}) \neq \emptyset$.

3.7 Evaluation of μ-Calculus Formulas on G^u

The techniques that we consider in this section have been presented in [CDHR06]. There they have been applied to game structures of incomplete information (cf. Section 2.8) with knowledge compatible parity conditions. We discuss the ideas and the properties of these techniques and we will see how we can apply them to our model where we restrict our attention to information compatible parity conditions as well. (In fact, the techniques cannot readily be applied to more general settings.)

The starting point is the "symbolic solution" of parity games, that means, the fact that for each $d < \omega$ there is a μ-calculus formula $\varphi_d \in L_\mu$ such that for each (deterministic, full information) parity game with at most d colors, φ_d describes the winning region of player 1. This formula prescribes a "dynamic program" for computing the winning region of player 1. If we consider a parity game with partial information, then of course the formula is evaluated on the corresponding game with full information.

Now the important observation on this is the following. If we use the universal powerset construction from Section 3.2.1 where the set of positions is downward closed, then the formulas of a syntactic fragment of the μ-calculus, which is still expressive enough to "solve" parity games, evaluate to downward closed sets. Since downward closed sets can be represented by their maximal elements, we can evaluate the formula on the lattice of antichains instead of the lattice of subsets. So if we use reasonable implementations of the involved operators, then this computation can be faster in practice, since in some sense it processes compressed data. The task here is of course to define appropriate operators for this evaluation.

Now we introduce the μ-calculus L_μ. Since we deal with the general k- dimensional μ-calculus in later sections, we introduce the full concept already here. The μ-calculus has been introduced by Kozen in [Koz83]. The multidimensional framework is due to Otto, [Ott99]. First we need the notion of a Kripke-structure and a little background on fixed points.

Definition 3.3. A *Kripke-structure* with actions from a set A and atomic propositions over a set I is a structure $\mathfrak{A} = (V, (E_a)_{a\in A}, (P_i)_{i\in I})$ over the signature $\tau = \{E_a \,|\, a \in A\} \cup \{P_i \,|\, i \in I\}$ where $E_a \subseteq V \times V$ for each $a \in A$ and $P_i \subseteq V$ for each $i \in I$.

Fixed Points. Let $(L, \sqsubseteq)$ be a lattice, that means, L is a set and $\sqsubseteq$ is a partial ordering on L such that for all $x, y \in L$ the set $\{x, y\}$ has an infimum $x \wedge y$ and a supremum $x \vee y$. The lattice is called complete, if each subset $L' \subseteq L$ has an infimum $\bigwedge L'$ and a supremum $\bigvee L'$. A function $F : L \to L$ is called monotone, if for all $x, y \in L$ with $x \sqsubseteq y$ we have $F(x) \sqsubseteq F(y)$. If F is monotone and $(L, \sqsubseteq)$ is complete, then F has a least fixed point, denoted by $\mathrm{lfp}(F)$ and a greatest fixed point, denoted by $\mathrm{gfp}(F)$ (Theorem of Knaster and Tarski). Now let x^α, $y^\alpha \in L$ for each ordinal number α be defined as follows.

- $x^0 = \bigwedge L$ and $y^0 = \bigvee L$.
- $x^{\alpha+1} = F(x^\alpha)$ and $y^{\alpha+1} = F(y^\alpha)$ for each ordinal number α.
- $x^\lambda := \bigvee\{x^\alpha \,|\, \alpha < \lambda\}$ and $y^\lambda := \bigwedge\{y^\alpha \,|\, \alpha < \lambda\}$ for each limit ordinal λ.

Then there are ordinal numbers β_x and β_y such that $x^{\beta_x} = x^\alpha$ for all $\alpha \geq \beta_x$ and $y^{\beta_y} = y^\alpha$ for all $\alpha \geq \beta_y$ and we have $x^{\beta_x} = \mathrm{lfp}(F)$ and $y^{\beta_y} = \mathrm{gfp}(F)$.

To define the logic L_μ^k we fix a $k < \omega$, a signature $\tau = \{E_a \,|\, a \in A\} \cup \{P_i \,|\, i \in I\}$ for some sets A and I and a k-tuple $\overline{x} = (x_1, \ldots, x_k)$ of variables. Furthermore we fix a set SV_k of k-ary relation symbols with $\mathrm{SV}_k \cap \tau = \emptyset$.

Syntax of L_μ^k. The formulas of the k-dimensional modal μ-calculus L_μ^k are generated by the following grammar, where $i \in I$, $1 \leq j \leq k$, $X \in \mathrm{SV}_k$, $a \in A$ and $\sigma \in \{1, \ldots, k\}^{\{1,\ldots k\}}$.

$$\varphi ::= P_{ij} \mid \neg P_{ij} \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle a \rangle_j \varphi \mid [a]_j \varphi \mid \varphi^\sigma \mid \mu X \varphi \mid \nu X \varphi$$

We say that an occurrence of a k-ary relation symbol $X \in \mathrm{SV}_k$ in a formula φ of L_μ^k is free, if it does not occur in the scope of a μ-operator or a ν-operator. The formula φ is called closed if it does not contain any free relation symbols from SV_k.

So in the logic L_μ^k we have atomic propositions P_i and the modal operators $\langle a \rangle$ and $[a]$ which always address a single component j of $\overline{x}$. Furthermore, we have closure under Boolean connectives and we have the k-ary variables $X \in \mathrm{SV}_k$. If $|\mathrm{SV}_k| = n$ for some $n < \omega$ then the above grammar defines the n-variable fragment $L_\mu^{k,n}$ of L_μ^k, that means, the syntactic fragment of L_μ^k where in each formula there occur at most n distinct variables. The formulas of the form P_{ij} and X are called atomic formulas. Furthermore we have variable substitutions, which allow us to substitute the values of some variables in $\overline{x}$ by the values of other variables. Finally, we have the fixed point operators μ and ν. The formulas from L_μ^k are evaluated on Kripke-structures. The precise semantics is the following.

Semantics of L_μ^k. Fix a Kripke-structure $\mathfrak{A} = (V, (E_a)_{a\in A}, (P_i)_{i\in I})$ over the signature τ. A valuation for the variables is a function $\mathcal{E} : \mathrm{SV}_k \to 2^{V^k}$ which maps every k-ary variable to a subset of V^k. For a k-ary variable $X \in \mathrm{SV}_k$ and a set $W \subseteq V^k$, by $\mathcal{E}[X \mapsto W]$ we denote the valuation which coincides with $\mathcal{E}$ except for mapping X to W.

Now a formula $\varphi \in L_\mu^k$ and a valuation $\mathcal{E}$ for the variables specify a subset $[\![\varphi]\!]_{\mathcal{E}}^{\mathfrak{A}}$ of V^k (the set of tuples from V^k where φ holds if the free variables are interpreted according to $\mathcal{E}$), which is defined inductively as follows.

- **atomic formulas**
  $[\![P_{ij}]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid v_j \in P_i\}$
  $[\![X]\!]_{\mathcal{E}}^{\mathfrak{A}} = \mathcal{E}(X)$
- **Boolean connectives**
  $[\![\neg\psi]\!]_{\mathcal{E}}^{\mathfrak{A}} = V^k \setminus [\![\psi]\!]_{\mathcal{E}}^{\mathfrak{A}}$
  $[\![\psi_1 \wedge \psi_2]\!]_{\mathcal{E}}^{\mathfrak{A}} = [\![\psi_1]\!]_{\mathcal{E}}^{\mathfrak{A}} \cap [\![\psi_2]\!]_{\mathcal{E}}^{\mathfrak{A}}$
  $[\![\psi_1 \vee \psi_2]\!]_{\mathcal{E}}^{\mathfrak{A}} = [\![\psi_1]\!]_{\mathcal{E}}^{\mathfrak{A}} \cup [\![\psi_2]\!]_{\mathcal{E}}^{\mathfrak{A}}$
- **modalities**
  $[\![\langle a\rangle_j\psi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid \ \exists\, w \in V : (v_j, w) \in E_a \wedge (\ldots, v_{j-1}, w, v_{j+1}, \ldots) \in [\![\psi]\!]_{\mathcal{E}}^{\mathfrak{A}}\ \}$
  $[\![[a]_j\psi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid \ \forall\, w \in V : (v_j, w) \in E_a \Rightarrow (\ldots, v_{j-1}, w, v_{j+1}, \ldots) \in [\![\psi]\!]_{\mathcal{E}}^{\mathfrak{A}}\ \}$
- **variable substitutions**
  $[\![\varphi^\sigma]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid (v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \in [\![\varphi]\!]_{\mathcal{E}}^{\mathfrak{A}}\}.$
- **fixed points**
  $[\![\mu X\varphi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \bigcap\{U \subseteq V^k \mid [\![\varphi]\!]_{\mathcal{E}[X\mapsto U]}^{\mathfrak{A}} = U\}$
  $[\![\nu X\varphi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \bigcup\{U \subseteq V^k \mid [\![\varphi]\!]_{\mathcal{E}[X\mapsto U]}^{\mathfrak{A}} = U\}$

So the formulas $\mu X\varphi$ and $\nu X\varphi$ evaluate to the least and greatest fixed point of the operator $U \mapsto [\![\varphi]\!]_{\mathcal{E}[X\mapsto U]}^{\mathfrak{A}}$, respectively. Since for each formula $\varphi \in L_\mu$, this operator is monotone and $(2^V, \subseteq)$ is a complete lattice, these fixed points exist.

We write $(\mathfrak{A}, \mathcal{E}, \overline{v}) \models \varphi$, if $\overline{v} \in [\![\varphi]\!]_{\mathcal{E}}^{\mathfrak{A}}$. If φ is a closed μ-calculus formula and $\mathfrak{A}$ is a Kripke-structure then we write $[\![\varphi]\!]^{\mathfrak{A}}$ for the evaluation of φ in $\mathfrak{A}$ (which is then of course independent of any valuation of variables).

3.7.1 μ-Calculus and Parity Games

Now we have a look at the μ-calculus formulas φ_d for $d < \omega$ which "solve" parity games with at most d colors and at the idea of the proof for the correctness.

Model Checking of L_μ. The abstract model checking problem asks, given a mathematical structure $\mathfrak{A}$ and a formula φ of some logic L which talks about $\mathfrak{A}$, whether

$\mathfrak{A} \models \varphi$. A generic idea to solve this problem is to reduce it to the strategy problem of a corresponding model checking game $G(\mathfrak{A}, \varphi)$. This idea has been applied for instance to first order logic FO, to basic modal logic ML and to the modal μ-calculus L_μ which we are interested in. (Recently the idea has also been applied to the quantitative μ-calculus, cf. [FGK08].)

Now we describe the model checking game for L_μ. We consider games without actions, that means, games where a player chooses a next position instead of an action. This doesn't matter as long as we consider *deterministic* games with *full* information and *position based* winning condition. Then we can just consider the corresponding game which is played on the underlying game graph by choosing positions instead of actions. Vice versa we can label an edge (u, v) in a game graph without actions by the action v. Notice that in this setting we just have the modal operators $\langle\,\rangle$ and $[\,]$. Nevertheless, the model checking games are defined for the multi-modal setting.

Consider a Kripke-structure $\mathfrak{A} = (V, (E_a)_{a\in A}, (P_i)_{i\in I})$ over the signature $\tau = \{E_a \,|\, a \in A\} \cup \{P_i \,|\, i \in I\}$ and a closed formula $\varphi \in L_\mu(\tau)$. We assume that for each variable $X \in \mathrm{SV}$ there is at most one occurrence of an operator λX for some $\lambda \in \{\mu, \nu\}$ in φ. First notice that we can always rename the variables in a formula to obtain an equivalent formula for which this assumption holds.

However, this renaming is not always w.l.o.g. If we are interested in the number of distinct variables which are needed to express certain properties in the μ-calculus, then for the fragment of L_μ where no variables are "reused", the fact that the *variable hierarchy* is strict follows from the fact that the *alternation hierarchy* is strict. (Since to construct a formula of alternation depth k, then, we need at least k distinct variables.) But in the very contrary, for any $d < \omega$ the formula φ_d which we define in this section is equivalent to a formula with only *two* distinct variables, but is *not* equivalent to any formula with alternation depth $\leq d - 1$. (See [Ber05] and the references there.) That in fact, the variable hierarchy of the μ-calculus is strict has been shown in [Ber05].

The model checking game $G(\mathfrak{A}, \varphi)$ is played by two players, called verifier (player 1) and falsifier (player 0). The positions of the game are tuples (ψ, v) where ψ is a subformula of φ and $v \in V$. From such a position, verifier wants to prove that $\mathfrak{A}, v \models \psi$ and falsifier wants to prove that $\mathfrak{A}, v \not\models \psi$, that means, $\mathfrak{A}, v \models \neg\psi$. The moves of the game are defined as follows.

- Verifier:
  1. $(\psi_1 \vee \psi_2, v) \to (\psi_1, v)$ and $(\psi_1 \vee \psi_2, v) \to (\psi_2, v)$.
  2. $(\langle a\rangle\psi, v) \to (\psi, w)$ for $w \in vE_a$.
- Falsifier:
  1. $(\psi_1 \wedge \psi_2, v) \to (\psi_1, v)$ and $(\psi_1 \wedge \psi_2, v) \to (\psi_2, v)$.
  2. $([a]\psi, v) \to (\psi, w)$ for $w \in vE_a$.

Furthermore, from positions (P, v) and $(\neg P, v)$ for some atomic proposition P, the only possible moves are to the position itself, that means, we have a selfloop on each such position and no further edges. So it does not matter which player moves. So far we have introduced the model checking game for plain modal logic. The model checking game for L_μ is obtained from this game by adding the following moves.

- $(\lambda X\psi, v) \to (\psi, v)$.
- $(X, v) \to (\psi, v)$ if $\mu X\psi$ is a subformula of φ.

So from a fixed point definition we go to the formula which defines the fixed point and if at some point in the game we reach the fixed point variable, then we go back to this formula. Since there is always just a single move from such positions, again, it does not matter which player moves. This gives us the game graph of the game. The winning condition of the model checking game is a *parity* condition, so we have to define a coloring of the positions of the game. The most important colors are those of the positions (X, v) for some variable $X \in \mathrm{SV}$, so we first define the coloring of such positions. We say that the variable $Y \in \mathrm{SV}$ depends on the variable $X \in \mathrm{SV}$, if $\lambda Y\psi$ is a subformula of φ for some $\lambda \in \{\mu, \nu\}$ and some formula ψ such that X is free in ψ.

- Parity:
  - $\mathrm{col}((X, v))$ is *odd*, if $\nu X\psi$ is a subformula of φ for some formula ψ.
  - $\mathrm{col}((X, v))$ is *even*, if $\mu X\psi$ is a subformula of φ for some formula ψ.
- Priority:
  $\mathrm{col}((X, v)) \leq \mathrm{col}((Y, v))$ if Y depends on X.

Now consider the minimal $1 < d < \omega$ such that with colors $0, \ldots, d-1$ such a coloring of the positions (X, v) can be constructed. We color each position (P, v) for some atomic proposition P with 1, if $v \in P$ and with 0, if $v \notin P$. Positions $(\neg P, v)$ are colored completely analog. Finally, all positions that have not been colored yet get the color $d - 1$.

Theorem 3.24. ([Sti99]) *For each closed formula $\varphi \in L_\mu$, each Kripke-structure $\mathfrak{A}$ and each position v of $\mathfrak{A}$ we have $\mathfrak{A}, v \models \varphi$ if and only if verifier (player 1) has a winning strategy for the model checking game $G(\mathfrak{A}, \varphi)$ from initial position (φ, v).*

To define μ-calculus formulas which describe winning regions in parity games we have to say how a parity game should be a model of such a formula first, that means, how we represent parity games as Kripke-structures. For a parity game $G = (V, V_0, E, \mathrm{col})$ (with $E \subseteq V \times V$) let $\mathfrak{A}_G := (V, E, V_0, V_1, (P_i)_{i=0,\ldots,d})$, where $d = |\operatorname{im}(\mathrm{col})|$ and $P_i = \{v \in V \mid \mathrm{col}(v) = i\}$ for $i \in \{0, \ldots, d-1\}$. Now consider the following μ-calculus formula φ_d.

$$\varphi_d := \mu X_0 \nu X_1 \mu X_2 \ldots \lambda X_{d-1} \bigvee_{i=0}^{d-1} \; (P_i \wedge [(V_1 \wedge \langle\,\rangle X_i) \vee (V_0 \wedge [\,]X_i)]).$$

Theorem 3.25. ([EJ91]) *Let* $d < \omega$ *and let* $G = (V, V_0, E, \mathrm{col})$ *be a parity game with* $\mathrm{im}(\mathrm{col}) \subseteq \{0, \ldots, d-1\}$. *Then* $\mathrm{Win}_1^G = [\![\varphi_d]\!]^{\mathfrak{A}_G}$.

One possibility to prove this is the following. We construct the model checking game $G(\mathfrak{A}_G, \varphi_d)$ and observe that essentially ("up to stupid moves") this game coincides with the game G itself. So we have $v \in \mathrm{Win}_1^G$ if and only if player 1 has a winning strategy for the game $G(\mathfrak{A}_G, \varphi_d)$ from (φ_d, v). According to Theorem 3.24 this is equivalent to $v \in [\![\varphi_d]\!]^{\mathfrak{A}_G}$. This idea is taken from [Grä].

Now we want to have a closer look at this formula. The expression $(V_1 \wedge \langle\,\rangle X_i) \vee (V_0 \wedge [\,] X_i)$ defines the controllable predecessor operator $\mathrm{CPre}_1 : 2^V \to 2^V$ of player 1 which intuitively maps a set $X \subseteq V$ of positions to the set $\mathrm{CPre}_1(X) \subseteq V$ of positions from which player 1 can *force* the game into X with *exactly one move.* So to describe the winning region of player 1 in a parity games we do not need the full expressive power of the operators $\langle\,\rangle$ and $[\,]$ (and of the predicates V_0 and V_1) but we only need them to define the controllable predecessor operator for player 1.

This observation gives rise to the definition of the following syntactic fragment of the μ-calculus which we call L_μ^{par}. So we fix a $d < \omega$ and a set $\mathrm{SV} = \mathrm{SV}_1$ of set variables such that $P_i \notin \mathrm{SV}$ for all $i \in \{0, \ldots, d-1\}$. Now the formulas of L_μ^{par} are generated by the following grammar, where $i \in \{0, \ldots, d-1\}$ and $X \in \mathrm{SV}$.

$$\varphi ::= P_i \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathrm{CPre}(\varphi) \mid \mu X \varphi \mid \nu X \varphi$$

The semantics of L_μ^{par} is the same as for L_μ where $\mathrm{CPre}(\varphi) \equiv (V_1 \wedge \langle\,\rangle\varphi) \vee (V_0 \wedge [\,]\varphi)$, that means, $\mathrm{CPre}(\varphi)$ evaluates to $\mathrm{CPre}_1([\![\varphi]\!]^{\mathfrak{A}}_{\mathcal{E}})$. Notice that this operator is of course monotone and thus the fixed points that can be defined in this logic always exist.

An extension to the nondeterministic case. Now we want to apply the "symbolic solution" of parity games to the universal games with full information which result from information compatible parity games via the universal powerset construction. The point is that those games are nondeterministic while the above setting deals with deterministic games. Of course we could construct the player 0 determinization of the universal game with full information and then apply the formula φ_d to it. But we want to evaluate μ-calculus formulas on G^u without explicitly constructing this game and so, this is not the right way to deal with the nondeterminism. Instead, we define a new formula, φ_d^Δ, which defines the winning region of player 1 in each parity game which is the player 0 determinization of a nondeterministic parity game and which can be evaluated on the nondeterministic game without explicitly constructing the player 0 determinization.

The idea of the formula is the following. We consider the formula φ_d as an element of the syntactic fragment L_μ^{par} of the modal μ-calculus so that we do not have $\langle\,\rangle$ and $[\,]$ as operators but only the controllable predecessor CPre_1 for player 1. The crucial point is that the controllable predecessor for player 1 of a set X of positions of a nondeterministic game is not defined by saying "if player 1 moves, there is an edge leading to X and if player 0 moves, all edges lead to X" but rather by saying "if player 1 moves there is an action such that all edges which are labelled with this action lead to X and if player 0 moves then for all actions, all edges which are labelled with this action lead to X". Now the formula φ_d^{Δ} expresses exactly this on the player 0 *determinization* of a nondeterministic game.

$$\varphi_d^{\Delta} := \mu X_0 \nu X_1 \mu X_2 \dots \lambda X_{d-1} \bigvee\nolimits_{i=0}^{d-1} \ (P_i \wedge \mathrm{CPre}_1(X_i)),$$

with $\mathrm{CPre}_1(X_i) := (V_1 \wedge \langle\,\rangle[\,]X_i) \vee (V_0 \wedge [\,][\,]X_i)$ for $i \in \{0, \dots, d-1\}$.

Theorem 3.26. *Let $G = (V, V_0, (E_a)_{a \in A}, \mathrm{col})$ be a nondeterministic parity game with $\mathrm{im}(\mathrm{col}) \subseteq \{0, \dots, d-1\}$ and let H be the player 0 determinization of G. Then $\mathrm{Win}_1^H \cap V = \llbracket \varphi_d^{\Delta} \rrbracket^{\mathfrak{A}_H} \cap V$.*

Proof. We adapt the idea for the proof of Theorem 3.25. We have to show that for each $v_0 \in V$, player 1 has a winning strategy for H from v_0 if and only if $v_0 \in \llbracket \varphi_d^{\Delta} \rrbracket^{\mathfrak{A}_H}$. Now by Theorem 3.24 it suffices to show that for each $v_0 \in V$, player 1 has a winning strategy for H from v_0 if and only if he has a winning strategy for the model checking game $G(\mathfrak{A}_H, \varphi_d^{\Delta})$ from $(\varphi_d^{\Delta}, v_0)$.

So let $v_0 \in V$. W.l.o.g. we assume that a position (X_i, v) is colored with i in $G(\mathfrak{A}_H, \varphi_d^{\Delta})$. Now let $u \in V$ and let $k \in \{0, \dots, d-1, d\}$. We abbreviate $\psi_k := \lambda X_k \dots \lambda X_{d-1} \bigvee_{i=0}^{d-1} (P_i \wedge [(V_1 \wedge \langle\,\rangle[\,]X_i) \vee (V_0 \wedge [\,][\,]X_i)])$. Now in the model checking game $G(\mathfrak{A}_H, \varphi_d^{\Delta})$, the move structure from position (ψ_k, u) is as follows.

First there is a sequence $(\psi_k, u) \to (\psi_{k+1}, u) \to \dots \to (\psi_{d-1}, u) \to (\psi_d, u)$ of moves where at each position (ψ_j, u) there is exactly one move possible. Then player 1 chooses some $i \in \{0, \dots, d-1\}$ and the game proceeds to $P_i \wedge [(V_1 \wedge \langle\,\rangle[\,]X_i) \vee (V_0 \wedge [\,][\,]X_i)]$. If $i \neq \mathrm{col}(u)$, then player 0 wins by choosing P_i in the next move. If $i = \mathrm{col}(u)$ then the only rational choice for player 0 is $(V_1 \wedge \langle\,\rangle[\,]X_i) \vee (V_0 \wedge [\,][\,]X_i)]$. Now player 1 chooses some $\sigma \in \{0, 1\}$ and the game continues at $V_\sigma \wedge \dots$ If $u \notin V_\sigma$ then player 0 wins by choosing V_σ in the next move. If $u \in V_\sigma$ then the possibilities for the next two moves in $G(\mathfrak{A}_H, \varphi_d^{\Delta})$ correspond exactly to the possibilities for the next two moves from u in H. (Notice that if $u \in V_1$ and u is not a terminal position in the game G, then each successor position of u in H belongs to player 0. If u is a terminal position in the game G, then in H, there is exactly one move possible from u, so it doesn't matter which player moves at u.) So we end up in a position (X_i, w), where w is reachable from u in H by two moves. Now (X_i, w) is colored with i and the only possible next move in $G(\mathfrak{A}_H, \varphi_d^{\Delta})$ leads to (ψ_i, w).

So altogether, from a position (ψ_k, u), until a position (ψ_i, w) is reached, the only real choices (up to "stupid ones") are those of a successor v of u and a successor w of v. These choices are (essentially) done by the same players as in H. Moreover, the only color in the corresponding sequence of moves that we have to take into account is i.

This shows how we can transfer winning strategies for player 1 from one game to the other. We simply use the bijections $\{(((\langle\,\rangle[\,]X_i, u), u) \mid u \in V_1 \cap P_i\}$ for $i = 0, \dots, d-1$. If we proceed from $G(\mathfrak{A}_H, \varphi_d^{\Delta})$ to H then this already defines a positional strategy for player 1. Vice versa, the value of the (positional) strategy for player 1 for $G(\mathfrak{A}_H, \varphi_d^{\Delta})$ on all other positions of player 1 is clear from the above analysis. To show that this construction preserves the winning property of a strategy, for a play in one game we just consider the corresponding play in the other game as it is given by the above analysis. ☐

As we have already mentioned we do not want to construct the determinization of the nondeterministic parity games that we have to solve explicitly. So we evaluate formulas of L_μ^{par} on nondeterministic games directly, in the following way.

Let G be a nondeterministic parity game with colors $0, \dots, d-1$ for some $1 < d < \omega$, let T be the set of terminal positions in G and let $\varphi \in L_\mu^{\mathrm{par}}$. We define the evaluation $[\![\varphi]\!]_{\mathcal{E}}^{G}$ as follows. First,

- $[\![P_0]\!]_{\mathcal{E}}^{G} = \{v \in V \mid \mathrm{col}(v) = 0\} \cup T \cap V_1$,
- $[\![P_1]\!]_{\mathcal{E}}^{G} = \{v \in V \mid \mathrm{col}(v) = 1\} \cup T \cap V_0$ and
- $[\![P_i]\!]_{\mathcal{E}}^{G} = \{v \in V \mid \mathrm{col}(v) = i\} \setminus T$ for $i \in \{2, \dots, d-1\}$.

Now the controllable predecessor operator is defined by

- $[\![\mathrm{CPre}(\varphi)]\!]_{\mathcal{E}}^{G} :=$

$$\begin{aligned} &\{u \in V_1 \setminus T \mid \ \exists a \in \mathrm{act}(u)\ \forall (u,v) \in E_a : v \in [\![\varphi]\!]_{\mathcal{E}}^{G}\ \} \\ \cup\ &\{u \in V_0 \setminus T \mid \ \forall a \in \mathrm{act}(u)\ \forall (u,v) \in E_a : v \in [\![\varphi]\!]_{\mathcal{E}}^{G}\ \} \\ \cup\ &\{u \in T \mid \ u \in [\![\varphi]\!]_{\mathcal{E}}^{G}\ \}. \end{aligned}$$

For all the other cases the evaluation $[\![\varphi]\!]_{\mathcal{E}}^{G}$ is defined in the obvious way. Together with Theorem 3.26, the following proposition now yields that with this evaluation, we can define the winning region of player 1 in each nondeterministic parity game by a formula of L_μ^{par}.

Proposition 3.21. *Let $G = (V, V_0, (E_a)_{a \in A}, \mathrm{col})$ be a nondeterministic parity game with colors from $\{0, \dots, d-1\}$ for some $1 < d < \omega$ and let H be the player 0 determinization of G. Furthermore let $\varphi \in L_\mu^{\mathrm{par}}$ with $\mathrm{CPre}_1(\psi) := (V_1 \wedge \langle\,\rangle[\,]\psi) \vee (V_0 \wedge [\,][\,]\psi)$ for $\psi \in L_\mu^{\mathrm{par}}$ and let $\mathcal{E}$ and $\mathcal{F}$ be valuations of the variables into G and H respectively, such that for each variable X we have $\mathcal{F}(X) \cap V = \mathcal{E}(X)$. Then $[\![\varphi]\!]_{\mathcal{F}}^{\mathfrak{A}_H} \cap V = [\![\varphi]\!]_{\mathcal{E}}^{G}$.*

It is worth noting that we can define the predicates P_i for $0 \leq i \leq d-1$ and the operator CPre as we have defined them for nondeterministic games in the (multi-) modal logic as well. So we can describe the winning region of player 1 in each nondeterministic parity game with a formula from the usual (multi-modal) μ-calculus. As the definitions are quite long, the above way for proving the correctness of our solution is much more pleasent.

Let $G = (V, V_0, (E_a)_{a \in A}, \mathrm{col})$ be a nondeterministic parity game with colors $0, \dots, d-1$ for some $1 < d < \omega$. We define $\mathfrak{A}_G := (V, (E_a)_{a \in A}, V_0, V_1, (P_i)_{i=0,\dots,d})$ where $P_i = \{v \in V \mid \mathrm{col}(v) = i\}$ for $i \in \{0, \dots, d-1\}$. Now for $\varphi \in L_\mu$ let

$$\mathrm{CPre}(\varphi) := (\bigvee_{a \in A} \langle a \rangle 1 \wedge [(V_1 \wedge \bigvee_{a \in A} (\langle a \rangle 1 \wedge [a]\varphi)) \vee (V_0 \wedge \bigwedge_{a \in A} (\langle a \rangle 1 \rightarrow [a]\varphi))]) \vee \bigwedge_{a \in A} [a]0 \wedge \varphi$$

and let the formulas col_i for $i \in \{0, \dots, d-1\}$ be defined as follows.

- $\mathrm{col}_0 := P_0 \vee \bigwedge_{a \in A} [a]0 \wedge V_1$.

- $\mathrm{col}_1 := P_1 \vee \bigwedge_{a \in A} [a]0 \wedge V_0$.
- $\mathrm{col}_i := P_i \wedge \bigvee_{a \in A} \langle a \rangle 1 \quad$ for $i \in \{0, \ldots, d-1\}$.

Then for each formula $\varphi \in L_\mu^{\mathrm{par}}$ and each valuation $\mathcal{E}$ of the variables into V we have $[\![\varphi(P_0/\mathrm{col}_0, \ldots, P_{d-1}/\mathrm{col}_{d-1})]\!]_{\mathcal{E}}^{\mathfrak{A}_G} = [\![\varphi]\!]_{\mathcal{E}}^{G}$, where $\varphi(P_0/\mathrm{col}_0, \ldots, P_{d-1}/\mathrm{col}_{d-1})$ is obtained from φ by replacing each occurrence of P_i for $i = 0, \ldots, d-1$ with the formula col_i.

3.7.2 Evaluation on G^u

From Theorem 3.8, Theorem 3.26, Proposition 3.21 and Proposition 2.3 we obtain the following result.

Theorem 3.27. *Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_1^A))$ with $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$ be an information compatible parity game and let $G^u = (V^u, V_0^u, (E_a^u)_{a \in A^u}, \mathrm{col}^u)$ be the corresponding universal game with full information. Furthermore, assume that for each $\pi \in P_{\mathrm{fin}}(V)$ the position $v(\pi) \in V^u$ is a nonterminal position. Then* $\mathrm{Win}_1^{\mathcal{G}} = \{v \in V \mid \{v\} \in [\![\varphi_d^\Delta]\!]^{G^u}\}$.

Now we shall see that if we evaluate formulas of L_μ^{par} on games which result from the universal powerset construction, then the sets to which the formulas evaluate are downward closed. Where a subset $Y \subseteq X$ of some ordered set $(X, \sqsubseteq)$ is called downward closed, if for all $y \in Y$ and all $x \in X$ with $x \sqsubseteq y$ we have $x \in Y$. The downward closure of Y is the set $Y\!\downarrow := \{x \in X \mid \exists\, y \in Y : x \sqsubseteq y\}$.

Let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$ be an information compatible parity game with $\mathrm{act}([u]_{\sim_1}) \neq \emptyset$ for all $u \in V_1$ and consider the corresponding universal game $G^u = (V^u, V_0^u, (E_a^u)_{a \in A^u}, \mathrm{col}^u)$ with full information. Then there are no terminal positions in G^u.

A valuation $\mathcal{E} : \mathrm{SV} \to 2^{V^u}$ of the variables is called downward closed if for all variables $X \in \mathrm{SV}$, the set $\mathcal{E}(X)$ is downward closed.

Proposition 3.22. *Let $\varphi \in L_\mu^{\mathrm{par}}$ and let $\mathcal{E}$ be a downward closed valuation of the variables. Then $[\![\varphi]\!]_{\mathcal{E}}^{G^u}$ is downward closed.*

Proof. We proceed by induction over the structure of φ. First, all the sets $[\![P_i]\!]_{\mathcal{E}}^{G^u}$ are obviously downward closed and each set $[\![X]\!]_{\mathcal{E}}^{G^u} = \mathcal{E}(X)$ for $X \in \mathrm{SV}$ is downward closed by assumption. If $[\![\varphi_1]\!]_{\mathcal{E}}^{G^u}$ and $[\![\varphi_2]\!]_{\mathcal{E}}^{G^u}$ are downward closed for $\varphi_1, \varphi_2 \in L_\mu^{\mathrm{par}}$ then obviously $[\![\varphi_1 \vee \varphi_2]\!]_{\mathcal{E}}^{G^u} = [\![\varphi_1]\!]_{\mathcal{E}}^{G^u} \cup [\![\varphi_2]\!]_{\mathcal{E}}^{G^u}$ and $[\![\varphi_1 \wedge \varphi_2]\!]_{\mathcal{E}}^{G^u} = [\![\varphi_1]\!]_{\mathcal{E}}^{G^u} \cap [\![\varphi_2]\!]_{\mathcal{E}}^{G^u}$ are downward closed as well. Now let $\varphi = \mathrm{CPre}(\psi)$, let $S \in [\![\varphi]\!]_{\mathcal{E}}^{G^u}$ and let $T \subseteq S$.

If $T \in V_1^u$, then $S \in V_1^u$ and since $S \in [\![\varphi]\!]_{\mathcal{E}}^{G^u}$, there is some action $a \in \mathrm{act}(S) \subseteq \mathrm{act}(T)$ such that for all $(S, U) \in E_a^u$ we have $U \in [\![\psi]\!]_{\mathcal{E}}^{G^u}$. Now let $(T, W) \in E_a^u$, that means, there is some some $v \in \mathrm{Post}_{[a]_{\sim_1}}(T)$ such that $W = \mathrm{Post}_{[a]_{\sim_1}}(T) \cap [v]_{\sim_1}$ and consider the set $U = \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1} \supseteq W$. By definition of E_a^u we have $(S, U) \in E_a^u$ and so $U \in [\![\psi]\!]_{\mathcal{E}}^{G^u}$. Now using $W \subseteq U$ and the induction hypothesis we get $W \in [\![\psi]\!]_{\mathcal{E}}^{G^u}$ and thus, $T \in [\![\varphi]\!]_{\mathcal{E}}^{G^u}$.

If $T \in V_0^u$, then $S \in V_0^u$ and since $S \in [\![\varphi]\!]^{G^u}_{\mathcal{E}}$, for each action $a \in \mathrm{act}(S)$ and all $(S, U) \in E_a^u$ we have $U \in [\![\psi]\!]^{G^u}_{\mathcal{E}}$. Now let $a \in \mathrm{act}(T)$ and let $(T, W) \in E_a^u$, that means, there is some some $v \in \mathrm{Post}_{[a]_{\sim_1}}(T)$ such that $W = \mathrm{Post}_{[a]_{\sim_1}}(T) \cap [v]_{\sim_1}$ and consider the set $U = \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [v]_{\sim_1} \supseteq W$. By definition of E_a^u we have $(S, U) \in E_a^u$ and so $U \in [\![\psi]\!]^{G^u}_{\mathcal{E}}$. Now using $W \subseteq U$ and the induction hypothesis we get $W \in [\![\psi]\!]^{G^u}_{\mathcal{E}}$ and thus, $T \in [\![\varphi]\!]^{G^u}_{\mathcal{E}}$.

Finally let $\varphi = \mu X \psi$ for some formula $\psi \in L_\mu^{\mathrm{par}}$ and some variable $X \in \mathrm{SV}$. (The reasoning if $\varphi = \nu X \psi$ is completely analog.) Let $x^0 = \emptyset$, $x^{\alpha+1} = [\![\psi]\!]^{G^u}_{\mathcal{E}[X \mapsto x^\alpha]}$ for each ordinal number α and $x^\lambda = \bigcup \{x^\alpha \mid \alpha < \lambda\}$ for each limit ordinal λ. Now by induction over α, using the induction hypothesis for ψ, we see that x^α is downward closed for each ordinal number α and thus, $[\![\varphi]\!]^{G^u}_{\mathcal{E}}$ is downward closed. ☐

Remark. The assumption that there are no terminal position in G^u is not just a technical simplification but it is an essential prerequisite for the fact that the formulas of L_μ^{par} evaluate to downward closed sets. (Notice that if $S \in V_1^u$ is a terminal position in G^u and $T \subseteq S$, then T is not necessarily a terminal position.)

Now any downward closed subset $\emptyset \neq Y \subseteq X$ of an ordered set $(X, \sqsubseteq)$ is completely determined by the set $\lceil Y \rceil = \{x \in Y \mid \neg \exists y \in Y : x \sqsubset y\}$ of its maximal elements, since $Y = \lceil Y \rceil \!\downarrow$. Notice that by definition, $\lceil Y \rceil$ is an antichain and since $\lceil Y \rceil = Y$ for each antichain Y, we have that $\{\lceil Y \rceil \mid \emptyset \neq Y \subseteq X\}$ is the set of antichains in $(X, \sqsubseteq)$. Where an antichain in $(X, \sqsubseteq)$ is a set $Y \subseteq X$ such that each two different elements from Y are not comparable with respect to $\sqsubseteq$. We want to use this compact representation of the sets $[\![\varphi]\!]^{G^u}_{\mathcal{E}}$ to evaluate formulas from L_μ^{par} on the lattice $(\mathcal{A}, \sqsubseteq)$, where $\mathcal{A}$ is the set of antichains in $(V^u, \subseteq)$ and $\sqsubseteq$ is defined as follows. For $q, q' \in \mathcal{A}$ let $q \sqsubseteq q'$, if for all $S \in q$ there is some $S' \in q'$ such that $S \subseteq S'$.

Proposition 3.23. *$(\mathcal{A}, \sqsubseteq)$ is a complete lattice and the following propositions hold.*

(1) $\emptyset$ is the least element and $\{[u]_{\sim_1} \mid u \in V\}$ is the greatest element of $(\mathcal{A}, \sqsubseteq)$.
(2) $\bigvee Q = \lceil \bigcup Q \rceil$ for all $Q \subseteq \mathcal{A}$.
(3) $\bigwedge Q = \lceil \{\bigcap q \mid q \subseteq V^u : q \cap q' \neq \emptyset \ \forall q' \in Q\} \rceil \setminus \{\emptyset\}$ for all $Q \subseteq \mathcal{A}$.

Now we have to define appropriate operations with which we can evaluate a formula of L_μ^{par} on the lattice $(\mathcal{A}, \sqsubseteq)$. For a formula $\varphi \in L_\mu^{\mathrm{par}}$ and a valuation $\mathcal{E}$ of the variables into V^u, such that for each variable X we have $\mathcal{E}(X) \in \mathcal{A}$, we denote the evaluation of φ on the lattice of antichains with respect to $\mathcal{E}$ by $[\![\varphi]\!]^{\mathcal{A}}_{\mathcal{E}}$. Most of the operations are already obvious. The most interesting one is of course the controllable predecessor operator CPre_1 of player 1. If we evaluate formulas on $(2^{V^u}, \subseteq)$ as usual, then the controllable predecessor of a set $q \subseteq V^u$ is the set of all positions in V^u from which player 1 can force the game into q with exactly one move. Now if q is downward closed and we represent it by the set $\lceil q \rceil$ of its maximal elements, then the controllable predecessor of $\lceil q \rceil$ must be the set of all positions in V^u from which player 1 can force the game into some subset of an element of $\lceil q \rceil$ with exactly one move. (And then we represent this set again by the set of its maximal elements.) Formally, for $q \in \mathcal{A}$ we define

$$\mathrm{CPre}_1^{\mathcal{A}}(q) := \lceil \mathrm{CPre}_1^{\subseteq}(q) \rceil$$

where $\mathrm{CPre}_1^{\subseteq}(q)$ is the union of the following two sets.

- $\{ S \in V_1^u \mid \exists a \in \mathrm{act}(S)\ \forall (S,T) \in E_a^u\ \exists U \in q : T \subseteq U \}$.
- $\{ S \in V_0^u \mid \forall a \in \mathrm{act}(S)\ \forall (S,T) \in E_a^u\ \exists U \in q : T \subseteq U \}$.

Proposition 3.24. $\mathrm{CPre}_1^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ *is a monotone operator.*

Now let $\mathcal{E}$ be an arbitrary valuation of the variables such that $\mathcal{E}(X) \in \mathcal{A}$ for each variable X. We evaluate formulas on the lattice of antichains as follows. Since according to Proposition 3.24 the operator $\mathrm{CPre}_1^{\mathcal{A}}$ is monotone, all the fixed points that can be defined actually exist.

- $\llbracket P_i \rrbracket_{\mathcal{E}}^{\mathcal{A}} = \{ P_i \cap [v]_{\sim_1} \mid v \in V,\ [v]_{\sim_1} \subseteq P_i \}$.
- $\llbracket X \rrbracket_{\mathcal{E}}^{\mathcal{A}} = \mathcal{E}(X)$.
- $\llbracket \varphi_1 \vee / \wedge \varphi_2 \rrbracket_{\mathcal{E}}^{\mathcal{A}} = \llbracket \varphi_1 \rrbracket_{\mathcal{E}}^{\mathcal{A}} \vee / \wedge \llbracket \varphi_2 \rrbracket_{\mathcal{E}}^{\mathcal{A}}$,
- $\llbracket \mathrm{CPre}(\psi) \rrbracket_{\mathcal{E}}^{\mathcal{A}} = \mathrm{CPre}_1^{\mathcal{A}}(\llbracket \psi \rrbracket_{\mathcal{E}}^{\mathcal{A}})$.
- $\llbracket \mu/\nu X \psi \rrbracket_{\mathcal{E}}^{\mathcal{A}} = \bigwedge / \bigvee \{ q \in \mathcal{A} \mid q = \llbracket \psi \rrbracket_{\mathcal{E}[X \mapsto q]}^{\mathcal{A}} \}$.

To show that with these operators, formulas of L_μ^{par} are evaluated on the lattice of antichains in the right way, we need the following result.

Proposition 3.25. *(1) For all* $Q \subseteq 2^{V^u}$ *we have* $\lceil \bigcup Q \rceil = \bigvee \{ \lceil q \rceil \mid q \in Q \}$.
(2) If $Q \subseteq 2^{V^u}$ *and all* $q \in Q$ *are downward closed, then* $\lceil \bigcap Q \rceil = \bigwedge \{ \lceil q \rceil \mid q \in Q \}$.

Now for a valuation $\mathcal{E}$ of the variables into V^u, let $\lceil \mathcal{E} \rceil$ denote the valuation of variables into V^u with $\lceil \mathcal{E} \rceil(X) = \lceil \mathcal{E}(X) \rceil$ for all variables X. Then the following proposition holds which shows that we have defined the operators for the evaluation of formulas $\varphi \in L_\mu^{\mathrm{par}}$ on the lattice of antichains in the right way.

Proposition 3.26. *Let* $\varphi \in L_\mu^{\mathrm{par}}$ *and let* $\mathcal{E}$ *be a downward closed valuation of the variables. Then* $\lceil \llbracket \varphi \rrbracket_{\mathcal{E}}^{G^u} \rceil = \llbracket \varphi \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}}$.

Proof. We proceed by induction over the structure of φ. First let $\varphi = P_i$ for some $i \in \{0, \dots, d-1\}$. Then $\lceil \llbracket \varphi \rrbracket_{\mathcal{E}}^{G^u} \rceil = \lceil \{ S \in V^u \mid S \subseteq P_i \} \rceil = \{ P_i \cap [v]_{\sim_1} \mid v \in V, [v]_{\sim_1} \subseteq P_i \} = \llbracket \varphi \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}}$. If $\varphi = X$ for some $X \in \mathrm{SV}$ then $\lceil \llbracket X \rrbracket_{\mathcal{E}}^{G^u} \rceil = \lceil \mathcal{E}(X) \rceil = \lceil \mathcal{E} \rceil(X) = \llbracket \varphi \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}}$. If $\varphi = \varphi_1 \vee \varphi_2$ for some formulas $\varphi_1, \varphi_2 \in L_\mu^{\mathrm{par}}$, then by induction hypothesis and Proposition 3.25 we have $\lceil \llbracket \varphi \rrbracket_{\mathcal{E}}^{G^u} \rceil = \lceil \llbracket \varphi_1 \rrbracket_{\mathcal{E}}^{G^u} \cup \llbracket \varphi_2 \rrbracket_{\mathcal{E}}^{G^u} \rceil = \lceil \llbracket \varphi_1 \rrbracket_{\mathcal{E}}^{G^u} \rceil \vee \lceil \llbracket \varphi_2 \rrbracket_{\mathcal{E}}^{G^u} \rceil = \llbracket \varphi_1 \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}} \vee \llbracket \varphi_2 \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}} = \llbracket \varphi \rrbracket_{\lceil \mathcal{E} \rceil}^{\mathcal{A}}$. The reasoning for the case where $\varphi = \varphi_1 \wedge \varphi_2$ is completely analog. Just notice that according to Proposition 3.22, the sets $\llbracket \varphi_i \rrbracket_{\mathcal{E}}^{G^u}$ for $i = 1, 2$ are downward closed.

Now let $\varphi = \mathrm{CPre}(\psi)$ for some formula $\psi \in L_\mu^{\mathrm{par}}$ and let first $S \in \lceil \llbracket \varphi \rrbracket_{\mathcal{E}}^{G^u} \rceil = \lceil \mathrm{CPre}_1(\llbracket \psi \rrbracket_{\mathcal{E}}^{G^u}) \rceil$. Then from S, player 1 can force the game into $\llbracket \psi \rrbracket_{\mathcal{E}}^{G^u}$ with exactly one move and so, from S, player 1 can force the game into subsets of elements from $q := \lceil [\![\psi]\!]^{G^u}_{\mathcal{E}} \rceil = [\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$ with exactly one move, that means, $S \in \mathrm{CPre}^{\subseteq}_1(q)$. Now we have to show that S is maximal in this set. So assume that S is not maximal, that means, there is some set $T \in \mathrm{CPre}^{\subseteq}_1(q)$ such that $S \subsetneq T$. Then from T player 1 can force the game into subsets of elements from q with exactly one move. But since according to Proposition 3.22, the set $[\![\psi]\!]^{G^u}_{\mathcal{E}}$ is downward closed, this yields that from T, player 1 can force the game into this set with exactly one move which contradicts $S \in \lceil \mathrm{CPre}_1([\![\psi]\!]^{G^u}_{\mathcal{E}}) \rceil$. Thus we have $\lceil [\![\varphi]\!]^{G^u}_{\mathcal{E}} \rceil \subseteq \lceil \mathrm{CPre}^{\subseteq}_1(q) \rceil = \mathrm{CPre}^{\mathcal{A}}_1(q) = \mathrm{CPre}^{\mathcal{A}}_1([\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}) = [\![\varphi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$. The proof of the converse inclusion uses exactly the same arguments.

Finally let $\varphi = \mu X \psi$ for some $\psi \in L^{\mathrm{par}}_\mu$ and some $X \in \mathrm{SV}$. Let $x^0 = \emptyset$, $x^{\alpha+1} = [\![\psi]\!]^{G^u}_{\mathcal{E}[X \mapsto x^\alpha]}$ for each ordinal number α and $x^\lambda = \bigcup\{x^\alpha \,|\, \alpha < \lambda\}$ for each limit ordinal λ. Using Proposition 3.22, a simple induction over α yields, that all the sets x^α are downward closed. Furthermore let $y^0 = \emptyset$, $y^{\alpha+1} = [\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil[X \mapsto y^\alpha]}$ for each ordinal number α and $y^\lambda = \bigvee\{y^\alpha \,|\, \alpha < \lambda\}$ for each limit ordinal λ. Clearly, $\lceil x^0 \rceil = \emptyset = y^0$. Now let α be an ordinal number. Since x^α is downward closed, the induction hypothesis for ψ and α yields $\lceil x^{\alpha+1} \rceil = \lceil [\![\psi]\!]^{G^u}_{\mathcal{E}[X \mapsto x^\alpha]} \rceil = [\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}[X \mapsto x^\alpha]\rceil} = [\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}[X \mapsto y^\alpha]\rceil} = [\![\psi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil[X \mapsto y^\alpha]} = y^{\alpha+1}$. Finally, if λ is some limit ordinal, then according to Proposition 3.25 we have $\lceil x^\lambda \rceil = \lceil \bigcup\{x^\alpha \,|\, \alpha < \lambda\} \rceil = \bigvee\{\lceil x^\alpha \rceil \,|\, \alpha < \lambda\} = \bigvee\{y^\alpha \,|\, \alpha < \lambda\} = y^\lambda$. Now there is some ordinal number β, such that $[\![\varphi]\!]^{G^u}_{\mathcal{E}} = x^\beta$ and $[\![\varphi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}]\!]} = y^\beta$ and so $\lceil [\![\varphi]\!]^{G^u}_{\mathcal{E}} \rceil = \lceil x^\beta \rceil = y^\beta = [\![\varphi]\!]^{\mathcal{A}}_{\lceil\mathcal{E}\rceil}$.

The reasoning if $\varphi = \nu X \psi$ is completely analog. Just notice that all the sets x^α are downward closed. □

From Theorem 3.27, Proposition 3.22 and proposition 3.26 we now obtain the main result of this section.

Theorem 3.28. *Let* $\mathcal{G} = (G, (\mathrm{vis}^V_i), (\mathrm{vis}^A_i))$ *with* $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$ *be an information compatible parity game such that* $\mathrm{act}([u]_{\sim_1}) \neq \emptyset$ *for all* $u \in V_1$. *Furthermore let* $G^u = (V^u, V^u_0, (E^u_a)_{a \in A^u}, \mathrm{col}^u)$ *be the corresponding universal game with full information and let* $\mathcal{A} = \{\lceil W \rceil \,|\, W \subseteq V^u\}$.

$$\textit{Then } \mathrm{Win}^{\mathcal{G}}_1 = \{v \in V \,|\; \{\{v\}\} \sqsubseteq [\![\varphi^\Delta_d]\!]^{\mathcal{A}}\}.$$

Chapter 4

Logical Definability

The main question in this chapter is, whether for a given logic, we can define the (global) relations $\sim_1^*$ and $\sim_1^+$ in this logic from the relation $\sim_1$ on $P_{\text{fin}}(v_0) \times P_{\text{fin}}(v_0)$ which is defined by $\pi \sim_1 \pi'$ if $\pi = \pi' = v_0$ or $|\pi| > 1$ and $|\pi'| > 1$ and $\text{last}_2(\pi) \sim_1 \text{last}_2(\pi')$ and from the edge relation E of (the unravelling of) the underlying game graph. We shall see that for LFP and GSO, the answer to this question is yes, while for the (bidirectional) two-dimensional μ-calculus and MSO, the answer is no.

Analyzing and (at least partially) answering this question has two intents. First, it helps us in understanding the objects that we have defined. For example if we could define the relation in first order logic, then the relations could be defined without using a recursion mechanism. Or if we could define them in the (bidirectional) two-dimensional μ-calculus, then they would be invariant under (bidirectional, multidimensional) bisimulation. And so on.

Second, once we have answered this question for certain logical systems like LFP and GSO, we can ask for the properties of the model that we get by considering an arbitrary formula from LFP or GSO, which defines a global equivalence relation. Of course we can also consider L_μ^2, MSO or FO. For example in first-order logic we can define the equivalence relations where a player can distinguish finite prefixes only by the last n positions for some fixed $n \in \mathbb{N}$ or where finite prefixes may only be indistinguishable if their length is at most n for some fixed $n \in \mathbb{N}$.

First we formulate this question precisely. Since we do not need the winning conditions of the games that we are considering in this chapter, we omit this component in the description of a game.

In the following, with $\text{Str}(\tau)$ we denote the class of all τ- structures for a signature τ. A class $\mathcal{K} \subseteq \text{Str}(\tau)$ is called closed under isomorphisms if for all $\mathfrak{A} \in \text{Str}(\tau)$ with $\mathfrak{A} \cong \mathfrak{B}$ for some $\mathfrak{B} \in \mathcal{K}$ we have $\mathfrak{A} \in \mathcal{K}$. For a class $\mathcal{K} \subseteq \text{Str}(\tau)$, by $\mathcal{K}^{\cong}$ we denote the closure of $\mathcal{K}$ under isomorphisms, that means, $\mathcal{K}^{\cong} = \{\mathfrak{A} \in \text{Str}(\tau) \mid \exists \mathfrak{B} \in \mathcal{K} : \mathfrak{A} \cong \mathfrak{B} \ \}$ is the smallest subclass of $\text{Str}(\tau)$ with $\mathcal{K} \subseteq \mathcal{K}^{\cong}$ such that $\mathcal{K}^{\cong}$ is closed under isomorphisms. We assume that the reader is familiar with syntax and semantics of plain first and second order logic.

Now we define the following two signatures.

($*$) $\tau^* := \{E, \sim\}$.

(+) $\tau^+ := \{E_0, E, \sim\}$.

Furthermore, for a game $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$, $G = (V, V_0, (f_a)_{a\in A})$ with partial information and a position $v_0 \in V$ we define the τ^*-structure $\mathfrak{A}^*_{\mathcal{G},v_0}$ and the τ^+-structure $\mathfrak{A}^+_{\mathcal{G},v_0}$ as follows.

(∗) $\mathfrak{A}^*_{\mathcal{G},v_0} := (P_{\mathrm{fin}}(v_0), E', \sim_1)$.

(+) $\mathfrak{A}^+_{\mathcal{G},v_0} := (P_{\mathrm{fin}}(v_0), E'_0, E', \sim_1)$.

- $E' = \{(\pi, \pi a v) \mid \pi, \pi a v \in P_{\mathrm{fin}}(v_0)\}$.
- $E'_0 = \{(\pi, \pi a v) \mid \pi, \pi a v \in P_{\mathrm{fin}}(v_0),\ \mathrm{last}(\pi) \in V_0 \text{ and } \mathrm{last}(\pi) \sim_1^V v\}$.
- $\sim_1 = \{(\pi, \pi') \mid \pi, \pi' \in P_{\mathrm{fin}}(v_0) \setminus \{v_0\},\ \mathrm{last}_2(\pi) \sim_1 \mathrm{last}_2(\pi')\} \cup \{(v_0, v_0)\}$.

Essentially, $(P_{\mathrm{fin}}(v_0), E')$ is the unravelling of G from v_0. But notice that we do not have separate edge relations E_a for $a \in A$ available but only their (disjoint) union. Now we define the following two classes of structures.

(∗) $\mathcal{K}^* := \{\mathfrak{A}^*_{\mathcal{G},v_0} \mid \mathcal{G} \text{ game with partial information, } v_0 \in V\}^{\cong}$.

(+) $\mathcal{K}^+ := \{\mathfrak{A}^+_{\mathcal{G},v_0} \mid \mathcal{G} \text{ game with partial information, } v_0 \in V\}^{\cong}$.

Remark. In this chapter we relax the requirement that each position in a game has at least one successor position. This avoids technical overhead in the proofs of the results for first-order logic and monadic second order logic. Nevertheless, we will suggest modifications of the proofs, which would make them work under the requirement that there are no terminal positions in a game.

The Question. Given $\# \in \{*, +\}$ and a logical language $\mathcal{L}$ over the signature $\tau^\#$, is there a formula $\varphi(x, y) \in \mathcal{L}$ (with two free element variables x and y) such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$,

$$\mathfrak{A}^\#_{\mathcal{G},v_0} \models \varphi(\pi, \pi') \iff \pi \sim_1^\# \pi' \ ?$$

Now we want to justify why we are using different signatures for the definition of $\sim_1^*$ and $\sim_1^+$. Intuitively, to define $\sim_1^*$ we just have to iterate $\sim_1$ as we have defined it above in a certain sense. (For the precise meaning of this, see Section 4.2.) So the signature τ^* suffices to define this relation. But to define $\sim_1^+$ we have to talk about private moves of player 0, which is not possible using the signature τ^*. Formally, we show that the (global) relation $\sim_1^+$ is not invariant under isomorphisms between structures from $\mathcal{K}^*$. So clearly, τ^* is not the right signature for defining $\sim_1^*$.

For this purpose we define the games G_0 and G_1 as follows. $G_0 = (\{0\}, \{0\}, f_{\circlearrowleft})$ with $f_{\circlearrowleft}(0) = 0$ and $G_1 = (\{0\}, \emptyset, f_{\circlearrowleft})$ with $f_{\circlearrowleft}(0) = 0$. Now consider corresponding games $\mathcal{G}_0$ and $\mathcal{G}_1$ with partial information (which are defined in the obvious way). Then we have $\mathfrak{A}^*_{\mathcal{G}_0,0} \cong \mathfrak{A}^*_{\mathcal{G}_1,0}$ but in the former structure there is only a single equivalence class with respect to $\sim_1^+$ while in the latter there are infinitely many such equivalence classes.

Remark. Instead of making the private moves of player 0 directly accessible via the relation E_0' we also could have made them just definable, for example by separating the edge relation into a move relation E_0'' containing *all* the moves of player 0 and an edge relation E_1'' containing all the moves of player 1 and furthermore putting a relation $\sim^V$ into the signature where $\pi \sim^V \pi'$ if and only if $\mathrm{last}(\pi) \sim_1^V \mathrm{last}(\pi')$.

4.1 First-Order Definability

First we show that for plain first-order logic we get a negative answer to our question. For this purpose we will use the so called Ehrenfeucht-Fraïssé-games, so we have a short look at these games first.

4.1.1 Elementary Equivalence

For a formula $\varphi \in \mathrm{FO}$, by $\mathrm{qr}(\varphi)$ we denote the quantifier rank of φ.

Definition 4.1. Let τ be a signature, let $\mathfrak{A}$, $\mathfrak{B}$ be τ-structures and let $n < \omega$.

(1) $\mathfrak{A}$ and $\mathfrak{B}$ are called *n-equivalent*, denoted by $\mathfrak{A} \equiv_n \mathfrak{B}$, if for all sentences $\varphi \in \mathrm{FO}(\tau)$ with $\mathrm{qr}(\varphi) \leq n$ we have $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$.
(2) $\mathfrak{A}$ and $\mathfrak{B}$ are called *elementary equivalent*, if $\mathfrak{A} \equiv_m \mathfrak{B}$ for all $m < \omega$.

The Ehrenfeucht-Fraïssé-game. Let τ be a relational signature and let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures. A partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is an injective function $f : \mathrm{dom}(f) \to B$ where $\mathrm{dom}(f) \subseteq A$, such that for all $n < \omega$, all n-ary relation symbols $R \in \tau$ and all $a_1, \dots, a_n \in \mathrm{dom}(f)$ we have

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \text{ if and only if } (f(a_1), \dots, f(a_n)) \in R^{\mathfrak{B}}.$$

Now let $m < \omega$. W.l.o.g. we assume that $A \cap B = \emptyset$. The Ehrenfeucht-Fraïssé-game $G_m(\mathfrak{A}, \mathfrak{B})$ is played by two players which are called spoiler and duplicator on positions from the sets

- $V_S = \{(a_1, \dots, a_i, b_1, \dots, b_i) \mid a_1, \dots, a_i \in A,\ b_1, \dots, b_i \in B,\ 0 \leq i \leq m\}$,
- $V_D^{\mathfrak{A}} = \{(a_1, \dots, a_{i+1}, b_1, \dots, b_i) \mid a_1, \dots, a_{i+1} \in A,\ b_1, \dots, b_i \in B,\ 0 \leq i < m\}$,
- $V_D^{\mathfrak{B}} = \{(a_1, \dots, a_i, b_1, \dots, b_{i+1}) \mid a_1, \dots, a_i \in A,\ b_1, \dots, b_{i+1} \in B,\ 0 \leq i < m\}$.

From a position $(a_1, \dots, a_i, b_1, \dots, b_i) \in V_S$ with $i < m$ the spoiler can move to all positions $(a_1, \dots, a_i, a_{i+1}, b_1, \dots, b_i) \in V_D^{\mathfrak{A}}$ and $(a_1, \dots, a_i, b_1, \dots, b_i, b_{i+1}) \in V_D^{\mathfrak{B}}$.

From positions $(a_1, \dots, a_i, a_{i+1}, b_1, \dots, b_i) \in V_D^{\mathfrak{A}}$ and $(a_1, \dots, a_i, b_1, \dots, b_i, b_{i+1}) \in V_D^{\mathfrak{B}}$ the duplicator can move to all positions $(a_1, \dots, a_i, a_{i+1}, b_1, \dots, b_i, b_{i+1}) \in V_S$.

A play π of $G_m(\mathfrak{A}, \mathfrak{B})$ starts at the initial position $\mathrm{first}(\pi) = \emptyset$ and takes m rounds so it ends in a position $\mathrm{last}(\pi) = (a_1, \dots, a_m, b_1, \dots, b_m) \in A^m \times B^m$. The play π is won by the duplicator if $\{(a_j, b_j) \mid j = 1, \dots, m\}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Otherwise the play π is won by the spoiler.

The subgame of $G_m(\mathfrak{A}, \mathfrak{B})$ from initial position $(a_1, \dots, a_i, b_1, \dots, b_i)$ for some $0 \leq i \leq m$ is denoted by $G_{m-i}(\mathfrak{A}, a_1, \dots, a_i, \mathfrak{B}, b_1, \dots, b_i)$.

The reason why we have introduced Ehrenfeucht-Fraïssé-games is that they form a powerful tool for proving or disproving elementary equivalence of structures. The theorem of Ehrenfeucht and Fraïssé precisely formulates in which way the games help us in doing so. A proof of the theorem can for example be found in [EFT07].

Theorem 4.1. *Let τ be a finite relational signature and let $\mathfrak{A}$, $\mathfrak{B}$ be τ-structures.*

(1) For every $m < \omega$ the duplicator has a winning strategy for the game $G_m(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A} \equiv_m \mathfrak{B}$.

(2) The duplicator has a winning strategy for the game $G_m(\mathfrak{A}, \mathfrak{B})$ for all $m < \omega$ if and only if $\mathfrak{A} \equiv \mathfrak{B}$.

4.1.2 Definability and Axiomatizability

Definition 4.2. Let τ be a signature and let $\mathcal{K} \subseteq \mathrm{Str}(\tau)$ be a class of τ-structures. A class $\mathcal{K}' \subseteq \mathcal{K}$ is called FO-*axiomatizable* in $\mathcal{K}$, if there is a set $\Phi \subseteq \mathrm{FO}(\tau)$ of $\mathrm{FO}(\tau)$-sentences such that for all structures $\mathfrak{A} \in \mathcal{K}$ we have $\mathfrak{A} \models \Phi$ if and only if $\mathfrak{A} \in \mathcal{K}'$. The class $\mathcal{K}'$ is called *finitely* FO-*axiomatizable* in $\mathcal{K}$, if Φ can be chosen finitely. In this case $\mathcal{K}'$ is axiomatizable in $\mathcal{K}$ by the $\mathrm{FO}(\tau)$-sentence $\varphi := \bigwedge \Phi$ and we say that $\mathcal{K}'$ is FO-*definable* in $\mathcal{K}$. If $\mathcal{K} = \mathrm{Str}(\tau)$ we simply say that $\mathcal{K}'$ is FO-axiomatizable (FO-definable).

Proposition 4.1. *Let τ be a signature and let $\mathcal{K}' \subseteq \mathcal{K} \subseteq \mathrm{Str}(\tau)$ be classes of τ-structures. If for each $n < \omega$ there are τ-structures $\mathfrak{A}_n \in \mathcal{K}'$ and $\mathfrak{B}_n \in \mathcal{K} \setminus \mathcal{K}'$ such that $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$, then the class $\mathcal{K}'$ is not* FO*-definable in $\mathcal{K}$.*

Now we define the class

$$\mathcal{K}_1^* := \{\mathfrak{A} \in \mathcal{K}^* \mid \exists\, \pi \in A \setminus \{v_0\}\ \forall\, \pi' \in A \setminus \{\pi\} : \pi' \not\sim_1^* \pi\ \}.$$

Intuitively, $\mathcal{K}_1^*$ captures the class of all games with partial information and some initial position v_0 such that after some finite prefix of length at least 2 of a play from initial position v_0, player 1 is certain about the complete play so far.

We want to have a short look at the fact that neither the class $\mathcal{K}^*$ nor the class $\mathcal{K}_1^*$ is FO-axiomatizable. For this purpose we use the compactness theorem. A proof of this theorem can for example be found in [EFT07].

Theorem 4.2. *(Compactness-Theorem for* FO*)*
Let τ be a signature and let $\Phi \subseteq \mathrm{FO}(\tau)$ be a set of τ-sentences. If each finite subset of Φ has a model, then Φ has a model as well.

Corollary 4.1. *The classes $\mathcal{K}^*$ and $\mathcal{K}_1^*$ are not* FO*-axiomatizable.*

Proof. Assume that $\Phi \subseteq \mathrm{FO}(\tau^*)$ axiomatizes $\mathcal{K}^*$ or $\mathcal{K}_1^*$. Let c be a constant-symbol and let $\varphi_0 := \forall x(\forall y(\neg Eyx) \rightarrow x \neq c \wedge \neg Exc)$ and $\varphi_n := \forall x(\forall y(\neg Eyx) \rightarrow \neg \exists x_1 \dots \exists x_n(Exx_1 \wedge \bigwedge_{i=1}^{n-1} Ex_ix_{i+1} \wedge Ex_nc))$ for $0 < n < \omega$.

Now let $\Phi' := \Phi \cup \{\varphi_n \,|\, n < \omega\}$ and let $\Phi'_0 \subseteq \Phi'$ be a finite subset of Φ'. Then $n = \max\,\{m < \omega \,|\, \varphi_m \in \Phi'_0\}$ exists, and we define $\mathfrak{A}_n = (A_n, E^{\mathfrak{A}}, \sim^{\mathfrak{A}}, c^{\mathfrak{A}})$ by $A_n = \{1, 2, 3, \dots, n+3\}$, $E^{\mathfrak{A}} = \{(i, i+1) \,|\, i \leq n+2\}$, $\sim^{\mathfrak{A}} = \mathrm{id}_{A_n}$ and $c^{\mathfrak{A}} = n+3$. Then we have $\mathfrak{A}_n \in \mathcal{K}^*_1 \subseteq \mathcal{K}^*$ and $\mathfrak{A}_n \models \varphi_m$ for all $m \leq n$. Thus we have $\mathfrak{A}_n \models \Phi'_0$ and so each finite subset of Φ' has a model. So the compactness Theorem for FO yields that Φ' has a model $\mathfrak{A} \models \Phi'$ as well.

Now since $\Phi \subseteq \Phi'$ we have $\mathfrak{A} \in \mathcal{K}^*$ and so in particular, $\mathfrak{A}$ is a rooted, directed tree with respect to the edge-relation $E^{\mathfrak{A}}$. So there is a finite $E^{\mathfrak{A}}$-path from the root of $\mathfrak{A}$ to $c^{\mathfrak{A}}$, say of length $n < \omega$. But this is a contradiction to $\mathfrak{A} \models \varphi_n$ and thus, the proof is complete. □

Now we show that $\mathcal{K}^*_1$ is not FO-definable in $\mathcal{K}^*$. For this purpose we define games $G_n = (V^n, V^n, f^n_l, f^n_r)$ and $H_n = (U^n, U^n, g^n_l, g^n_r)$ for $n < \omega$ as follows. We abbreviate $m := 2^{n+1} + 1$.

- $V^n = \{0\} \cup \{u_1, \dots, u_m\} \cup \{v_1, \dots, v_m\}$.
    - $f^n_l(0) = u_1$ and $f^n_l(u_i) = u_{i+1}$ for $i = 1, \dots, m-1$.
    - $f^n_r(0) = v_1$ and $f^n_r(v_i) = v_{i+1}$ for $i = 1, \dots, m-1$.
- $U^n = \{1\} \cup \{x_1, \dots, x_m, x_{m+1}\} \cup \{y_1, \dots, y_m\}$.
    - $g^n_l(1) = x_1$ and $g^n_l(x_i) = x_{i+1}$ for $i = 1, \dots, m$.
    - $g^n_r(1) = y_1$ and $g^n_r(y_i) = y_{i+1}$ for $i = 1, \dots, m-1$.

Now we define $\mathrm{vis}^V_1(v) = 0$ for all $v \in V^n$, $\mathrm{vis}^V_1(u) = 0$ for all $u \in U^n$ and $\mathrm{vis}^A_1(l) = \mathrm{vis}^A_1(r) = 0$ for both games. We call the corresponding games with partial information $\mathcal{G}_n$ and $\mathcal{H}_n$ and we abbreviate $\mathfrak{A}_n := \mathfrak{A}^*_{\mathcal{G}_n,0}$ and $\mathfrak{B}_n := \mathfrak{A}^*_{\mathcal{H}_n,1}$. The games are delineated in Figure 4.1.

Theorem 4.3. *For all $n < \omega$ we have $\mathfrak{B}_n \in \mathcal{K}^*_1$, $\mathfrak{A}_n \in \mathcal{K}^* \setminus \mathcal{K}^*_1$ and $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$.*

Proof. Let $n < \omega$ and let $m := 2^{n+1} + 1$. By definition, $\mathfrak{A}_n = (A_n, (\tau^*)^{\mathfrak{A}_n}) \in \mathcal{K}^*$ and $\mathfrak{B}_n = (B_n, (\tau^*)^{\mathfrak{B}_n}) \in \mathcal{K}^*$. Now for $\pi \in P_{\mathrm{fin}}(0) \setminus \{0\}$, $\pi = 0lu_1 \dots lu_k$ or $\pi = 0rv_1 \dots rv_k$ for some $1 \leq k \leq m$. Moreover, $\mathrm{vis}^*_1(0lu_1 \dots lu_k) = \mathrm{vis}^*_1(0rv_1 \dots rv_k)$ for all $1 \leq k \leq m$, and so, for each $\pi \in P_{\mathrm{fin}}(0) \setminus \{0\}$ there is some $\pi' \in P_{\mathrm{fin}}(0) \setminus \{\pi\}$ such that $\mathrm{vis}^*_1(\pi) = \mathrm{vis}^*_1(\pi')$. Thus, $\mathfrak{A}_n \notin \mathcal{K}^*_1$. Furthermore, there is no $\pi \in P_{\mathrm{fin}}(1)$ such that $\mathrm{vis}^*_1(\pi) = \mathrm{vis}^*_1(1lx_1lx_2l \dots x_mlx_{m+1})$, so $\mathfrak{B}_n \in \mathcal{K}^*_1$.

To prove that $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$, according to Theorem 4.1 we have to show that the duplicator has a winning strategy for the Ehrenfeucht-Fraïssé-game $G_n(\mathfrak{A}_n, \mathfrak{B}_n)$. For this purpose we show by induction over i that for all $i < n$, if after i turns of the game $G_n(\mathfrak{A}_n, \mathfrak{B}_n)$ the elements $a_1, \dots, a_i$ from $\mathfrak{A}_n$ and $b_1, \dots, b_i$ from $\mathfrak{B}_n$ are chosen, such that some invariant $(\mathrm{Inv})_i$ holds, then for each move of the spoiler, there is a move of the duplicator such that for the corresponding elements $a_1, \dots, a_i, a_{i+1}$ and $b_1, \dots, b_i, b_{i+1}$, $(\mathrm{Inv})_{i+1}$ holds. We choose $(\mathrm{Inv})_i$ to be as follows, where $l, k \in \{1, \dots, i\}$.

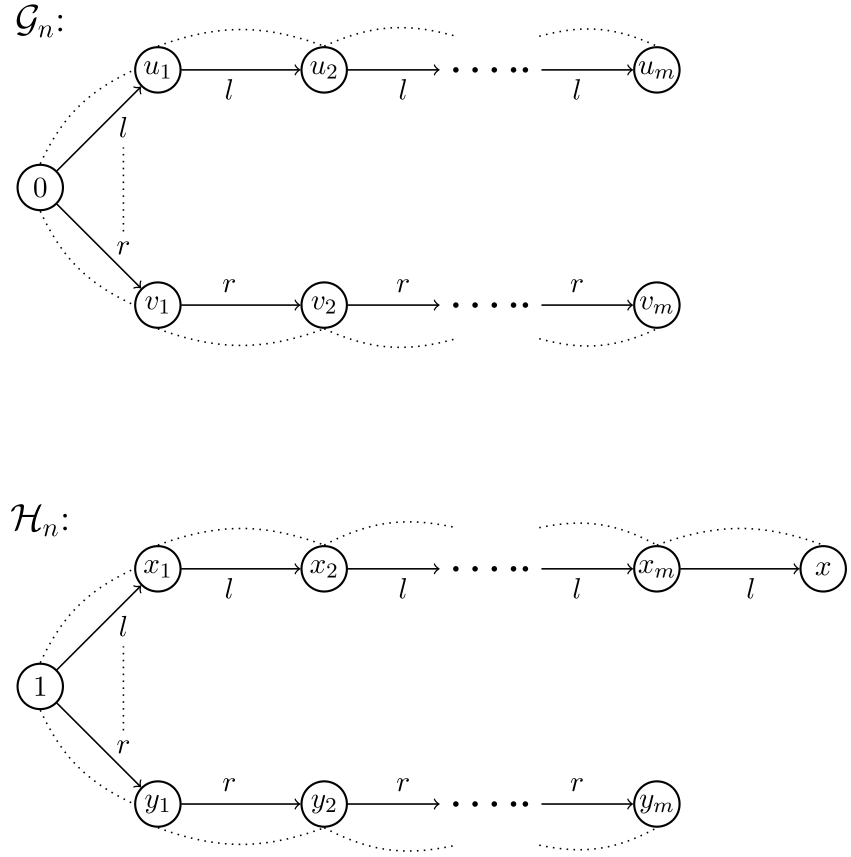

Figure 4.1: The games $\mathcal{G}_n$ and $\mathcal{H}_n$.

(I1) $d^+(a_l, a_k) =_{\zeta(i)} d^+(b_l, b_k)$.

(I2) $d^+(0, a_l) =_{\zeta(i)} d^+(1, b_l)$ and $d^+(a_l, u_m) = \infty$ if and only if $d^+(b_l, x_{m+1}) = \infty$.

(I3) $d^+(a_l, u_m) =_{\zeta(i)} d^+(b_l, x_{m+1})$ and $d^+(a_l, v_m) =_{\zeta(i)} d^+(b_l, y_m)$.

Where $\zeta(i) := 2^{n-i+1} - 1$ and $p =_k q$ for $p, q, k \in \omega \cup \{\infty\}$ if $p = q$ or $p > k$ and $q > k$. Furthermore, for $x, y \in A_n$ and $x, y \in B_n$, respectively, $d^+(x, y)$ is the directed distance from x to y with respect to the edge relation E', that means, $d^+(x, y)$ is the length of a shortest directed path from x to y with edges from E' if there is such a path, and $d^+(x, y) = \infty$, otherwise. Furthermore, in the following we denote the undirected distance between x and y with respect to E' by $d(x, y)$, that means, $d(x, y)$ is the length of a shortest path between x and y with edges from the symmetric closure of E' if there is such a path, and $d(x, y) = \infty$, otherwise. Notice that $d(x, y) \leq \min\{d^+(x, y), d^+(y, x)\}$. Furthermore, in our case we always have $d(x, y) < \infty$. Notice that in particular, if $(\mathrm{Inv})_i$ holds, then $a_j \mapsto b_j$, $j = 1, \ldots, i$ is a partial isomorphism from $\mathfrak{A}_n$ to $\mathfrak{B}_n$. Since this is unambiguous, in the following, we address an element π from A_n and B_n respectively, by $\mathrm{last}(\pi)$.

First let $i = 0$ and assume that the spoiler chooses an element $a_1 \in A_n$. If $d^+(0, a_1) \leq 2^n$ and $d^+(a_1, u_m) < \infty$, then the duplicator chooses the uniquely determined element $b_1 \in B_n$ with $d^+(1, b_1) = d^+(0, a_1)$ and $d^+(b_1, x_{m+1}) < \infty$. If $d^+(a_1, u_m) \leq 2^n$, then the duplicator chooses the uniquely determined element $b_1 \in B_n$ with $d^+(b_1, x_{m+1}) = d^+(a_1, u_m)$. In both cases, $(\mathrm{Inv})_1$ holds. If the spoiler chooses an element $a_1 \in A_n$ with $d^+(a_1, v_m) < \infty$, then the reasoning is completely analog. If the spoiler chooses an element $b_1 \in B_n$ with $d^+(b_1, y_m) < \infty$ or if he chooses an element $b_1 \in B_n$ with $d^+(b_1, x_{m+1}) < \infty$ and $d^+(1, b_1) \leq 2^n$ or $d^+(b_1, x_{m+1}) \leq 2^n$, then again we can apply the same reasoning as before. Now finally consider the case where the spoiler chooses the uniquely determined element $b_1 \in B_n$ with $d^+(1, b_1) > 2^n$, $d^+(b_1, x_{m+1}) < \infty$ and $d^+(b_1, x_{m+1}) > 2^n$. Then the duplicator chooses the uniquely determined element $a_1 \in A_n$ with $d^+(0, a_1) = 2^n$ and $d^+(a_1, u_m) < \infty$. Again, $(\mathrm{Inv})_1$ holds.

Now let $0 < i < n$ and let $f_i : \{a_1, \ldots, a_i, 0, u_m, v_m\} \to \{b_1, \ldots, b_i, 1, x_{m+1}, y_m\}$ be defined by $f_i(a_j) := b_j$ for $j = 1, \ldots, i$, $f_i(0) := 1$, $f_i(u_m) := x_{m+1}$ and $f_i(v_m) := y_m$. Then f_i is a partial isomorphism since due to $(\mathrm{Inv})_i$ we have $a_j = 0$ if and only if $b_j = 1$, $a_j = u_m$ if and only if $b_j = x_{m+1}$ and $a_j = v_m$ if and only if $b_j = y_m$ for all $j \in \{1, \ldots, i\}$. Now we have to distinguish certain cases. W.l.o.g. we assume that the spoiler chooses an element that has not yet been chosen.

(1) First let the spoiler choose an element $a_{i+1} \in A_n$ with $d^+(a_{i+1}, u_m) < \infty$.

Now consider an element $x \in \mathrm{dom}(f_i)$ which is closest to a_{i+1} with respect to the undirected distance d. Then $d^+(x, u_m) < \infty$ and so due to $(\mathrm{Inv})_i$ we have $d^+(f_i(x), x_{m+1}) < \infty$. If $a_{i+1} = 0$, the duplicator chooses $b_{i+1} = 1$. Then due to (I1) of $(\mathrm{Inv})_i$, $(\mathrm{Inv})_{i+1}$ holds as well. Now let $a_{i+1} \neq 0$.

(1.1) Consider the case $d(x, a_{i+1}) = d^+(x, a_{i+1}) \leq 2^{n-i}$.
First we define an element b_{i+1} which the duplicator chooses.

Since $d^+(x, u_m) =_{\zeta(i)} d^+(f_i(x), x_{m+1})$ there is a (uniquely determined) element $b_{i+1} \in B_n$ such that $d^+(f_i(x), b_{i+1}) = d^+(x, a_{i+1})$. Obviously $b_{i+1} \notin \{b_1, \ldots, b_i\}$, because otherwise we would have $d^+(f_i(x), b_l) = d^+(f_i(x), b_{i+1}) = d^+(x, a_{i+1}) \leq 2^{n-i} - 1$ for some $l \leq i$ and so due to $(\mathrm{Inv})_i$, $d^+(x, a_{i+1}) = d^+(f_i(x), b_l) = d^+(x, a_l)$ and thus $a_{i+1} = a_l$. Furthermore, $f_i(x)$ is the element from $\mathrm{im}(f_i)$ which is closest to b_{i+1} with respect to d. Because if there would be some element $b \in \mathrm{im}(f_i)$ which is closer to b_{i+1} than $f_i(x)$ with respect to d, then we would have $d^+(f_i(x), b) < 2^{n-i+1}$ and so due to $(\mathrm{Inv})_i$, $d^+(x, f_i^{-1}(b)) = d^+(f_i(x), b)$. But since $d^+(x, a_{i+1}) = d^+(f_i(x), b_{i+1})$ this would yield that $d(a_{i+1}, f^{-1}(b)) = d(b_{i+1}, b) < d(b_{i+1}, f_i(x)) = d(a_{i+1}, x)$ and so $f_i^{-1}(b)$ would be closer to a_{i+1} than x with respect to d.
Now we show that this choice of the duplicator preserves the invariant.

Let $a \in \mathrm{dom}(f_i)$. If $d^+(a, u_m) = \infty$ then $d^+(f_i(a), x_{m+1}) = \infty$ and so, $d^+(a, a_{i+1}) = d^+(a_{i+1}, a) = d^+(f_i(a), b_{i+1}) = d^+(b_{i+1}, f_i(a)) = \infty$. Now let $d^+(a, u_m) < \infty$ and thus, $d^+(f_i(a), x_{m+1}) < \infty$.

(1.1.1) Consider the case $d(a, a_{i+1}) = d^+(a, a_{i+1})$.

Then due to the choice of x, $d^+(a, a_{i+1}) = d^+(a, x) + d^+(x, a_{i+1})$.

If $d^+(a, x) \leq 2^{n-i+1} - 1$ then $(\mathrm{Inv})_i$ yields $d^+(a, x) = d^+(f_i(a), f_i(x))$ and so $d^+(f_i(a), b_{i+1}) = d^+(f_i(a), f_i(x)) + d^+(f_i(x), b_{i+1}) = d^+(a, a_{i+1})$.

If $d^+(a, x) > 2^{n-i+1} - 1$ then clearly $d^+(a, a_{i+1}) > 2^{n-i} - 1$ and due to $(\mathrm{Inv})_i$ we have $d^+(f_i(a), f_i(x)) > 2^{n-i+1} - 1$. So $d^+(f_i(a), b_{i+1}) = \infty$ or $d^+(f_i(a), b_{i+1}) = d^+(f_i(a), f_i(x)) + d^+(f_i(x), b_{i+1}) > 2^{n-i+1} - 1 > 2^{n-i} - 1$.

Furthermore, $d^+(a_{i+1}, a) = \infty$ and if $d^+(b_{i+1}, f_i(a)) \leq 2^{n-i} - 1$ then we would have $d^+(f_i(x), f_i(a)) = d^+(f_i(x), b_{i+1}) + d^+(b_{i+1}, f_i(a)) \leq 2^{n-i+1} - 1$. This implies $d^+(x, a) \leq 2^{n-i+1} - 1$ which is impossible since $d^+(x, a) = \infty$. Thus we have $d^+(b_{i+1}, f_i(a)) > 2^{n-i} - 1$.

(1.1.2) Now consider the case $d(a, a_{i+1}) = d^+(a_{i+1}, a)$.

Then $d^+(a_{i+1}, a) = d^+(x, a) - d^+(x, a_{i+1})$.

If $d^+(x, a) \leq 2^{n-i+1} - 1$ then due to $(\mathrm{Inv})_i$ we have $d^+(x, a) = d^+(f_i(x), f_i(a))$ and so $d^+(b_{i+1}, f_i(a)) = d^+(f_i(x), f_i(a)) - d^+(f_i(x), b_{i+1}) = d^+(a_{i+1}, a)$.

If $d^+(x, a) > 2^{n-i+1} - 1$ then due to $(\mathrm{Inv})_i$ we have $d^+(f_i(x), f_i(a)) > 2^{n-i+1} - 1$ and since $d^+(x, a_{i+1}) = d^+(f_i(x), b_{i+1}) \leq 2^{n-i}$ it follows that $d^+(a_{i+1}, a) = d^+(x, a) - d^+(x, a_{i+1}) > 2^{n-i} - 1$ and $d^+(b_{i+1}, f_i(a)) = \infty$ or $d^+(b_{i+1}, f_i(a)) = d^+(f_i(x), f_i(a)) - d^+(f_i(x), b_{i+1}) > 2^{n-i} - 1$.

Furthermore we have $d^+(a, a_{i+1}) = \infty$ and if $d^+(f_i(a), b_{i+1}) \leq 2^{n-i} - 1$ then, since $f_i(x)$ is the element from $\mathrm{im}(f_i)$ which is closest to b_{i+1} with respect to d, $d^+(f_i(a), f_i(x)) = d^+(f_i(a), b_{i+1}) - d^+(f_i(x), b_{i+1}) \leq 2^{n-i} - 1 + 2^{n-i} \leq 2^{n-i+1} - 1$. This implies $d^+(a, x) \leq 2^{n-i+1} - 1$ which is impossible, since $d^+(a, x) = \infty$. Thus we have $d^+(f_i(a), b_{i+1}) > 2^{n-i} - 1$.

(1.2) Now let $d(x, a_{i+1}) = d^+(a_{i+1}, x) \leq 2^{n-i}$.

Then since $d^+(0, x) =_{\zeta(i)} d^+(0, f_i(x))$, there is a (uniquely determined) element $b_{i+1} \in B_n$ with $d^+(b_{i+1}, f_i(x)) = d^+(a_{i+1}, x)$ which the duplicator chooses. With the same reasoning as for the case (1.1) we can show that this choice preserves the invariant.

(1.3) Let $d(x, a_{i+1}) = d^+(x, a_{i+1}) > 2^{n-i}$.

Then for the element $y \in \mathrm{dom}(f_i) \setminus \{x\}$ which is closest to x with respect to the directed distance d^+ we have $d^+(x, y) > 2 \cdot 2^{n-i} > 2^{n-i+1} - 1$, since x has been chosen as the element which is closest to a_{i+1} with respect to d. So by $(\mathrm{Inv})_i$ we have $d^+(f_i(x), z) > 2^{n-i+1} - 1$ where z is the element in $\mathrm{im}(f_i)$ which is closest to $f_i(x)$ with respect to d^+. So the duplicator can choose the element b_{i+1} in such a way that $d^+(f_i(x), b_{i+1}) > 2^{n-i} - 1$ and $d^+(b_{i+1}, z) > 2^{n-i} - 1$.

(1.4) Finally, let $d(x, a_{i+1}) = d^+(a_{i+1}, x) > 2^{n-i}$.

As in the case (1.3), the duplicator can choose the element b_{i+1} in such a way that $d^+(b_{i+1}, f_i(x)) > 2^{n-i} - 1$ and $d^+(z, b_{i+1}) > 2^{n-i} - 1$.

Clearly, in both of the cases (1.3) and (1.4), the invariant is preserved by the choice of the duplicator. This concludes the case (1), where the spoiler chooses an element a_{i+1} from A_n with $d^+(a_{i+1}, u_m) < \infty$. The reasoning in all the other cases is completely analog. □

Corollary 4.2. *The class $\mathcal{K}_1^*$ is not* FO*-definable in $\mathcal{K}^*$.*

Corollary 4.3. *There is no formula $\varphi(x, y) \in \mathrm{FO}(\tau^*)$, such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ we have $\mathfrak{A}^*_{\mathcal{G}, v_0} \models \varphi(\pi, \pi')$ if and only if $\pi \sim_1^* \pi'$.*

Proof. Assume that such a formula $\varphi(x, y)$ exists. We define $\varphi_{v_0}(x) := \neg\exists y(Eyx)$. Now consider the $\mathrm{FO}(\tau^*)$-sentence $\psi := \exists x(\neg\varphi_{v_0}(x) \wedge \forall y(\neg(x = y) \rightarrow \neg\varphi(x, y)))$. Obviously this sentence defines the class $\mathcal{K}_1^*$ in $\mathcal{K}^*$ which is a contradiction to Corollary 4.2. □

Remark. As we have mentioned, we allow terminal positions in games in this chapter, but we want to suggest a possibility to prove the previous result under the requirement that there are no terminal positions in a game. One can easily check that putting selfloops on the terminal positions in $\mathcal{G}_n$ and $\mathcal{H}_n$ would lead to games which are not appropriate. But we could add positions p to $\mathcal{G}_n$ and q to $\mathcal{H}_n$ and insert edges $u_m \to_l p$, $v_m \to_r p$, $p \to_{\circlearrowleft} p$, $x_{m+1} \to_l q$, $y_m \to_r q$ and $q \to_{\circlearrowleft} q$. Then we make the positions p and q distinguishable from all other positions for player 1, that means, $\mathrm{vis}_1^A(p) = p$ and $\mathrm{vis}_1^A(q) = q$. Using the resulting games we could show that $\mathcal{K}_1^*$ is not FO-definable in $\mathcal{K}^*$ just as we have done this using $\mathcal{G}_n$ and $\mathcal{H}_n$.

Now we shall see that the class $\mathcal{K}_1^*$ is FO-axiomatizable in $\mathcal{K}^*$ by an infinite set of FO(τ^*)-sentences. To show this, for a given game $\mathcal{G}$ with partial information and an initial node $v_0 \in V$, for each $n < \omega$ we consider the relation $\approx_1^n = \{\pi \sim_1^* \pi' \mid |\pi| = |\pi'| = n\}$. Then for all $n < \omega$ there is a FO(τ^*)-sentence $\varphi_n(x, y)$ such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\text{fin}}(v_0)$ we have $\mathfrak{A}^*_{\mathcal{G},v_0} \models \varphi_n(\pi, \pi')$ if and only if $\pi \approx_1^n \pi'$. The definition of φ_n is straightforward, where φ_{v_0} is as in the proof of Corollary 4.3.

$$\varphi_n(x, y) := \exists v_1 \ldots \exists v_n \exists w_1 \ldots \exists w_n$$

$$\Big(\varphi_{v_0}(v_1) \wedge \varphi_{v_0}(w_1) \wedge (x = v_n) \wedge (y = w_n) \wedge \bigwedge_{i=1}^{n-1} (Ev_iv_{i+1} \wedge Ew_iw_{i+1}) \wedge \bigwedge_{i=1}^{n} (v_i \sim w_i)\Big).$$

Furthermore for $n < \omega$ we define the FO(τ^*)-sentence ψ_n by

$$\psi_n := \exists x \Big(\exists v_1 \ldots \exists v_n (\varphi_{v_0}(v_1) \wedge (x = v_n) \wedge \bigwedge_{i=1}^{n-1} Ev_iv_{i+1}) \wedge \forall y((x \neq y) \to \neg\varphi_n(x, y))\Big).$$

Now the class $\mathcal{K}_1^*$ is axiomatized in $\mathcal{K}^*$ by $\Phi := \{\psi_n \mid n \geq 2\}$.

Now we want to modify our proof technique in order to show that $\sim_1^+$ is not FO-definable in $\mathcal{K}^+$ as well. Notice that to prove this we cannot use the same games as in the proof of Theorem 4.3. Since if we define

$$\mathcal{K}_1^+ := \{\mathfrak{A} \in \mathcal{K}^+ \mid \ \exists\, \pi \in A \setminus \{v_0\} \ \ \forall\, \pi' \in A \setminus \{\pi\} : \pi \not\sim_1^+ \pi' \ \},$$

then for the games $\mathcal{G}_n$ and $\mathcal{H}_n$ from the proof of Theorem 4.3 we have $\mathfrak{A}^+_{\mathcal{G}_n,0} \notin \mathcal{K}_1^+$ and $\mathfrak{A}^+_{\mathcal{H}_n,1} \notin \mathcal{K}_1^+$ for all $n < \omega$. (All the moves in both of the games $\mathcal{G}_n$ and $\mathcal{H}_n$ are private for player 0 and so we have $\pi \sim_1^+ v_0$ for all finite prefixes π of plays in $\mathcal{G}_n$ and $\mathcal{H}_n$.) So we have to use other structures.

We define the games $G_n = (V^n, \{0\}, f_l^n, f_r^n, f_\to^n)$ and $H_n = (U^n, \{1\}, g_l^n, g_r^n, f_\to^n)$ for $n < \omega$ as follows. We abbreviate $m := 2^{n+1} + 1$.

- $V^n = \{0\} \cup \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_m\}$.
- $f_l^n(0) = u_1$ and $f_r^n(0) = v_1$.
- $f_\to^n(u_i) = u_{i+1}$ and $f_\to^n(v_i) = v_{i+1}$ for $i = 1, \ldots, m-1$.
- $U^n = \{1\} \cup \{x_1, \ldots, x_m, x_{m+1}\} \cup \{y_1, \ldots, y_m\}$.

- $g_l^n(1) = x_1$ and $g_r^n(1) = y_1$.
- $f_{\rightarrow}^n(x_i) = x_{i+1}$ and $f_{\rightarrow}^n(y_i) = y_{i+1}$ for $i = 1, \dots, m-1$.
- $f_{\rightarrow}^n(x_m) = x_{m+1}$.

Now we define $\mathrm{vis}_1^V(0) = 0$, $\mathrm{vis}_1^V(v) = 1$ for all $v \in V^n$, $\mathrm{vis}_1^V(1) = 0$ and $\mathrm{vis}_1^V(u) = 1$ for all $u \in U^n$. Furthermore we define $\mathrm{vis}_1^A(l) = \mathrm{vis}_1^A(r) = 0$ and $\mathrm{vis}_1^A(\rightarrow) = \rightarrow$ for both games. We call the corresponding games with partial information $\mathcal{G}_n$ and $\mathcal{H}_n$ and we abbreviate $\mathfrak{A}_n := \mathfrak{A}^+_{\mathcal{G}_n,0}$ and $\mathfrak{B}_n := \mathfrak{A}^+_{\mathcal{H}_n,1}$.

With exactly the same arguments as in the proof of Theorem 4.3 and Corollary 4.3 we can show the following results.

Theorem 4.4. *For all $n < \omega$ we have $\mathfrak{B}_n \in \mathcal{K}_1^+$, $\mathfrak{A}_n \in \mathcal{K}^+ \setminus \mathcal{K}_1^+$ and $\mathfrak{A}_n \equiv_n \mathfrak{B}_n$.*

Corollary 4.4. *The class $\mathcal{K}_1^+$ is not* FO*-definable in $\mathcal{K}^+$.*

Corollary 4.5. *There is no formula $\varphi(x,y) \in \mathrm{FO}(\tau^+)$, such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ we have $\mathfrak{A}^+_{\mathcal{G},v_0} \models \varphi(\pi,\pi')$ if and only if $\pi \sim_1^+ \pi'$.*

4.2 Fixed Point Definability

4.2.1 Least Fixed Point Logic

In this section we show that for FO + fixed points, a formula as we have asked for in our question exists. For a little background on fixed points, see Section 3.7.

Syntax of LFP. Let τ be a signature and let RV be a set of relation variables, each of which has some fixed arity, such that $\tau \cap \mathrm{RV} = \emptyset$. A formula ψ is called positive in some relation symbol $R \in \mathrm{RV}$, if R does not occur within ψ under an odd number of negation symbols. $\mathrm{LFP}(\tau)$ is defined by the inductive clauses which define the syntax of $\mathrm{FO}(\tau)$ and the following additional rules.

- If $R \in \mathrm{RV}$ is a k-ary relation variable and $\overline{x}$ is a k-tuple of variables, then $R\overline{x}$ is an atomic formula.
- If $R \in \mathrm{RV}$ is a k-ary relation variable, ψ is a formula which is positive in R, $\overline{x}$ is a k-tuple of variables and $\overline{t}$ is a k-tuple of terms, then $[\mathrm{lfp}\, R\,\overline{x}\,\psi](\overline{t})$ and $[\mathrm{gfp}\, R\,\overline{x}\,\psi](\overline{t})$ are formulas.

Now let $R \in \mathrm{RV}$ be a k-ary relation symbol, let ψ be a formula which is positive in R and let $\overline{x}$ be a k-tuple of variables. Furthermore let $\mathcal{I}$ be a valuation of the first order variables and let $\mathcal{J}$ be a valuation of the second order variables, that means, $\mathcal{J}$ maps each relation symbol $P \in \mathrm{RV}$ to a relation $\mathcal{J}(P) \subseteq A^l$ where l is the arity of P. Now for a tuple $\overline{a} \in A^k$ and some relation $X \subseteq A^k$, by $\mathcal{I}[\overline{x} \mapsto \overline{a}]$ and $\mathcal{J}[R \mapsto X]$ we denote the valuations of variables which coincide with $\mathcal{I}$ and $\mathcal{J}$ except for mapping $\overline{x}$ to $\overline{a}$ and mapping R to X, respectively. Then for each τ-structure $\mathfrak{A}$, the following operator is monotone.

$$F_\psi : 2^{A^k} \to 2^{A^k},\ F_\psi(X) := \{\overline{a} \in A^k \mid \mathfrak{A}, \mathcal{I}[\overline{x} \mapsto \overline{a}], \mathcal{J}[R \mapsto X] \models \psi\}.$$

Semantics of LFP**.** Let τ be a signature and let RV be a set of relation variables, each of which has some fixed arity, such that $\tau \cap \mathrm{RV} = \emptyset$ and let $\mathfrak{A}$ be a τ-structure. Furthermore let $\mathcal{I}$ be a valuation of the first order variables and let $\mathcal{J}$ be a valuation of the second order variables. Now let $\overline{x}$ be a k-tuple of variables, let $\overline{t}$ be a k-tuple of terms and let ψ be a formula which is positive in R.

- $\mathfrak{A}, \mathcal{I}, \mathcal{J} \models [\mathrm{lfp}\, R\, \overline{x}\, \psi](\overline{t})$ if and only if $\overline{t}^{\mathfrak{A},\mathcal{I}} \in \mathrm{lfp}(F_\psi)$.
- $\mathfrak{A}, \mathcal{I}, \mathcal{J} \models [\mathrm{gfp}\, R\, \overline{x}\, \psi](\overline{t})$ if and only if $\overline{t}^{\mathfrak{A},\mathcal{I}} \in \mathrm{gfp}(F_\psi)$.

Now let $\varphi_{v_0}(x) := \neg\exists y(Eyx)$. Then the following formulas define the relations $\sim_1^*$ and $\sim_1^+$ respectively.

Fixed Point Formulas for $\sim_1^*$ and $\sim_1^+$.

$(*)$ $\varphi^*(x,y) := [\mathrm{gfp}\ \sim_1^*\ ab\, \varphi_f^*(a,b,\sim_1^*)](x,y)$ with

$$\varphi_f^*(a,b,R) := (\varphi_{v_0}(a) \wedge \varphi_{v_0}(b)) \vee (a \sim b \wedge \exists u \exists v(Eua \wedge Evb \wedge Ruv))$$

$(+)$ $\varphi^+(x,y) := [\mathrm{gfp}\ \sim_1^+\ ab\, \varphi_f^+(a,b,\sim_1^+)](x,y)$ with

$$\begin{aligned}\varphi_f^+(a,b,R) := {} & (\varphi_{v_0}(a) \wedge \varphi_{v_0}(b)) \vee (a \sim b \wedge \exists u \exists v(Eua \wedge Evb \wedge Ruv)) \vee \\ & \exists u(E_0ua \wedge Rub) \vee \exists v(E_0vb \wedge Rav)\end{aligned}$$

Proposition 4.2. *For each game $\mathcal{G}$ with partial information, each $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ the following propositions hold.*

(1) $\mathfrak{A}^*_{\mathcal{G},v_0} \models \varphi^*(\pi,\pi')$ if and only if $\pi \sim_1^* \pi'$.
(2) $\mathfrak{A}^+_{\mathcal{G},v_0} \models \varphi^+(\pi,\pi')$ if and only if $\pi \sim_1^+ \pi'$.

Proof. We prove only proposition (1). Proposition (2) can be proved with very similar arguments. Let $A = P_{\mathrm{fin}}(v_0)$ and let $F^* : 2^{A\times A} \to 2^{A\times A}$ be defined by $F^*(R) = \{(\pi,\pi') \in A \times A \mid \mathfrak{A}^*_{\mathcal{G},v_0} \models \varphi_f^*(\pi,\pi',R)\}$. Then F^* is monotone and for all $\pi,\pi' \in A$ we have $\mathfrak{A}^*_{\mathcal{G},v_0} \models \varphi^*(\pi,\pi')$ if and only if $(\pi,\pi') \in \mathrm{gfp}(F^*)$. Furthermore, we define $\sim^\alpha \subseteq A \times A$ for each ordinal α by $\sim^0 = A \times A$, $\sim^{\alpha+1} = F^*(\sim^\alpha)$ for each ordinal α and $\sim^\lambda = \bigcap\{\sim^\alpha \mid \alpha < \lambda\}$ for each limit ordinal λ. Now we show that $\sim_1^* = \bigcap\{\sim_1^n \mid n < \omega\}$. Since it is easy to see that $F^*(\sim_1^*) =\sim_1^*$, this yields $\sim_1^*= \mathrm{gfp}(F^*)$.

So let $\pi,\pi' \in P_{\mathrm{fin}}(v_0)$ and let first $\pi \sim_1^* \pi'$. By induction on $k := |\pi| = |\pi'|$ we show that $\pi \sim_1^n \pi'$ for all $n < \omega$. If $k = 1$ then $\pi = \pi' = v_0$ so by definition of $\sim_1^n$ we have $\pi \sim_1^n \pi'$ for all $n < \omega$. Now let $k > 1$. Then there are $\overline{\pi}, \overline{\pi}' \in P_{\mathrm{fin}}(v_0)$, $a, a' \in A$ and $v, v' \in V$ such that $\pi = \overline{\pi}av$ and $\pi' = \overline{\pi}'a'v'$. Since $\pi \sim_1^* \pi'$ we have $\overline{\pi} \sim_1^* \overline{\pi}'$ and

$av \sim_1 a'v'$. By induction hypothesis, $\overline{\pi} \sim_1^n \overline{\pi}'$ for all $n < \omega$ and we have $\pi \sim_1^0 \pi'$. So by definition of $\sim_1^{n+1}$ we have $\pi \sim_1^n \pi'$ for all $n < \omega$.

Now let $\pi \sim_1^n \pi'$ for all $n < \omega$. By induction on $k := |\pi|$ we show that $\pi \sim_1^* \pi'$. If $k = 1$, then $\pi = \pi' = v_0$, so let $k > 1$. Then clearly $|\pi'| > 1$ as well and so there are $\overline{\pi}, \overline{\pi}' \in P_{\text{fin}}(v_0)$, $a, a' \in A$ and $v, v' \in V$ such that $\pi = \overline{\pi}av$ and $\pi' = \overline{\pi}'a'v'$. Since $\pi \sim_1^n \pi'$ for all $n < \omega$, $\overline{\pi} \sim_1^n \overline{\pi}'$ for all $n < \omega$ must hold as well and so by induction hypothesis we have $\overline{\pi} \sim_1^* \overline{\pi}'$. Furthermore, $\pi \sim_1^n \pi'$ for all $n < \omega$ yields $\pi \sim_1 \pi'$, that means, $av \sim_1 a'v'$ and so $\pi \sim_1^* \pi'$. ☐

Now the operators lfp and gfp are dual, that means we have $[\text{gfp } R\overline{x}\,\psi](\overline{t}) \equiv \neg[\text{lfp } R\overline{x}\,\neg\psi[R/\neg R]](\overline{t})$ where $\psi[R/\neg R]$ is obtained from ψ by replacing each occurrence of R with $\neg R$. So the formula φ^* yields a formula which defines $\sim_1^*$ and where only the operator lfp is used. But notice that in the inductive evaluation of this formula, we compute the least fixed point of the dual operator of F^* (with F^* as in the proof of Proposition 4.2) and then we complement this fixed point. (Where the dual operator $\overline{F}^* : 2^{A^k} \to 2^{A^k}$ of F^* is defined by $\overline{F}^*(X) = A^k \setminus F^*(A^k \setminus X)$.) Indeed, the least and greatest fixed point of F^* *coincide*, and in the inductive construction of $\text{lfp}(F^*)$ we construct the relations $\approx_1^n = \{\pi \sim_1^* \pi' \,|\, l(\pi) = l(\pi') \leq n\}$ for $n < \omega$. (While in the inductive construction of $\text{lfp}(F^+)$ we only construct a subset of $\approx_1^{n,+} = \{\pi \sim_1^+ \pi' \,|\, l(\pi), l(\pi') \leq n\}$ in each step.)

$$(*)\quad \psi^*(x,y) := [\text{lfp } \sim_1^* ab\,\varphi_f^*(a,b,\sim_1^*)](x,y).$$

$$(+)\quad \psi^+(x,y) := [\text{lfp } \sim_1^+ ab\,\varphi_f^+(a,b,\sim_1^+)](x,y).$$

Proposition 4.3. *For each game $\mathcal{G}$ with partial information, each $v_0 \in V$ and all $\pi, \pi' \in P_{\text{fin}}(v_0)$ the following propositions hold.*

(1) $\mathfrak{A}^*_{\mathcal{G},v_0} \models \psi^*(\pi,\pi')$ if and only if $\pi \sim_1^* \pi'$.
(2) $\mathfrak{A}^+_{\mathcal{G},v_0} \models \psi^+(\pi,\pi')$ if and only if $\pi \sim_1^+ \pi'$.

4.2.2 Two-Dimensional μ-Calculus

In this section we again get a negative answer to our question, that means, we cannot find a two-dimensional μ-calculus formula which defines the relation $\sim_1^*$ (or the relation $\sim_1^+$ respectively), even if we add modalities which access predecessors to the language. The tool for showing this is bisimulation, so we have a short look at bisimulations between Kripke-structures first. (For the definition of the logic L_μ^2 and the notion of a Kripke-structure, see Section 3.7.)

Definition 4.3. Let $\mathfrak{A} = (V, (E_a)_{a\in A}, (P_i)_{i\in I})$ and $\mathfrak{A}' = (V', (E'_a)_{a\in A}, (P'_i)_{i\in I})$ be Kripke-structures over the signature $\tau = \{E_a \,|\, a \in A\} \cup \{P_i \,|\, i \in I\}$. A relation $B \subseteq V \times V'$ is called a *bisimulation* between $\mathfrak{A}$ and $\mathfrak{A}'$ if the following conditions hold for all $(v, v') \in B$.
atomic: $v \in P_i$ if and only if $v' \in P'_i$ for all $i \in I$.
forth: For all $a \in A$ and all $w \in V$ such that $(v,w) \in E_a$ there is some $w' \in V'$ such that $(v', w') \in E'_a$ and $(w, w') \in B$.

back: For all $a \in A$ and all $w' \in V'$ such that $(v', w') \in E_a$ there is some $w \in V$ such that $(v, w) \in E_a$ and $(w, w') \in B$.
For $v \in V$ and $v' \in V'$ we say that $(\mathfrak{A}, v)$ and $(\mathfrak{A}', v')$ are *bisimilar* if there is a bisimulation $B \subseteq V \times V'$ such that $(v, v') \in B$.

Now the following result tells us, that the logic L^k_μ is invariant under bisimulations. We use this property of L^k_μ to derive the non-definability of $\sim^*_1$ and $\sim^+_1$. A proof of the theorem can be found in [Ott99].

Theorem 4.5. *Let $\varphi \in L^k_\mu$ be a closed formula and let $\mathfrak{A}$, $\mathfrak{A}'$ be Kripke-structures. If $\overline{a} \in A^k$ and $\overline{a}' \in (A')^k$, such that $(\mathfrak{A}, a_j)$ and $(\mathfrak{A}', a'_j)$ are bisimilar for $j = 1, \ldots, k$, then $(\mathfrak{A}, \overline{a}) \models \varphi$ if and only if $(\mathfrak{A}', \overline{a}') \models \varphi$.*

Corollary 4.6. *Let $\# \in \{*, +\}$. Then there is no closed formula $\varphi \in L^2_\mu(\tau^\#)$, such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\text{fin}}(v_0)$ we have $(\mathfrak{A}^\#_{\mathcal{G},v_0}, (\pi, \pi')) \models \varphi$ if and only if $\pi \sim^\#_1 \pi'$.*

Proof. We define the game $G = (V^1, V^1, f_a, f_b)$ by $V^1 = \{0, 1, 2, 3, 4\}$, $f_a(0) = 1$, $f_b(0) = 2$, $f_a(1) = 3$, $f_a(2) = 4$, $f_a(3) = 3$ and $f_a(4) = 4$. Now consider corresponding games $\mathcal{G}$ and $\mathcal{H}$ with partial information where in both games we have $\text{vis}^A_1(a) = \text{vis}^A_1(b) = a$. Furthermore, in $\mathcal{G}$ we have $\text{vis}^V_1(0) = 0$, $\text{vis}^V_1(1) = \text{vis}^V_1(2) = 1$ and $\text{vis}^V_1(3) = \text{vis}^V_1(4) = 3$. Finally, in $\mathcal{H}$ we have $\text{vis}^V_1(0) = 0$, $\text{vis}^V_1(1) = 1$, $\text{vis}^V_1(2) = 2$ and $\text{vis}^V_1(3) = \text{vis}^V_1(4) = 3$. The games are depicted in Figure 4.2.

Now let $\pi := 0a1a3$ and $\pi' := 0a2a4$. Then obviously $(\mathfrak{A}^\#_{\mathcal{G},0}, \pi)$ and $(\mathfrak{A}^\#_{\mathcal{H},0}, \pi)$ as well as $(\mathfrak{A}^\#_{\mathcal{G},0}, \pi')$ and $(\mathfrak{A}^\#_{\mathcal{H},0}, \pi')$ are bisimilar, but $\text{vis}^\#_1(\pi) = 0a1a3 = \text{vis}^\#_1(\pi')$ in $\mathcal{G}$ and $\text{vis}^\#_1(\pi) = 0a1a3 \neq 0a2a4 = \text{vis}^\#_1(\pi')$ in $\mathcal{H}$. So using the bisimulation invariance of L^2_μ, the proof is finished. ☐

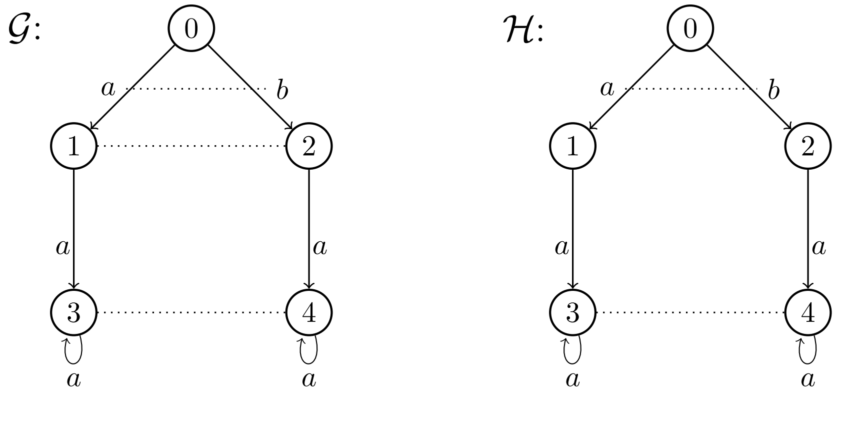

Figure 4.2: The games $\mathcal{G}$ and $\mathcal{H}$.

Remark. Theorem 4.5 has originally been formulated only for a single modal framework. (Cf. [Ott99].) Of course this is not appropriate for our concerns, since we are dealing with structures over the signatures $\tau^* = \{E, \sim\}$ and $\tau^+ = \{E, E_0, \sim\}$ where we have at least two different binary relations which of course have to be distinguished. Fortunately, according to the brief discussion in Section 2.4 of [Ott99], Theorem 4.5 (as well as the other results in [Ott99]) can be extended to the more general multi modal framework where arbitrarily many relations are allowed.

Of course, the previous result had to be expected since in the definition of $\sim_1^*$ and $\sim_1^+$ we refer to the predecessors of the positions under consideration and in the language L_μ^k the modalities only access the successors of the recent positions. So we should have a look at a two-dimensional μ-calculus where we have also modalities which access predecessors. For this purpose we extend the language L_μ^k to $(L_B)_\mu^k$ by adding the following rules to the grammar which defines the syntax of L_μ^k.

- **backwards modalities** If φ is a formula of $(L_B)_\mu^k$ then for all $1 \leq j \leq k$ and all $a \in A$, $\langle a_B \rangle_j \varphi$ and $[a_B]_j \varphi$ are formulas of $(L_B)_\mu^k$.

The semantics of the new operators is defined in the obvious way.

- **backwards modalities**
  $[\![\langle a_B \rangle_j \psi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid \exists w \in V : (w, v_j) \in E_a \wedge (\ldots, v_{j-1}, w, v_{j+1}, \ldots) \in [\![\psi]\!]_{\mathcal{E}}^{\mathfrak{A}} \}$
  $[\![[a_B]_j \psi]\!]_{\mathcal{E}}^{\mathfrak{A}} = \{\overline{v} \in V^k \mid \forall w \in V : (w, v_j) \in E_a \Rightarrow (\ldots, v_{j-1}, w, v_{j+1}, \ldots) \in [\![\psi]\!]_{\mathcal{E}}^{\mathfrak{A}} \}$

Of course, bisimulations are not appropriate for comparing Kripke-structures if we allow backwards modalities as well. To obtain an appropriate notion of bisimulation we have to regard the backwards modalities as well and thus, we introduce bidirectional bisimulations in the natural way.

Definition 4.4. Let $\mathfrak{A} = (V, (E_a)_{a \in A}, (P_i)_{i \in I})$ and $\mathfrak{A}' = (V', (E'_a)_{a \in A}, (P'_i)_{i \in I})$ be Kripke-structures over the signature $\tau = \{E_a \mid a \in A\} \cup \{P_i \mid i \in I\}$. A bisimulation $B \subseteq V \times V'$ between $\mathfrak{A}$ and $\mathfrak{A}'$ is called *bidirectional* if the following conditions hold for all $(v, v') \in B$.
forth backwards: For all $a \in A$ and all $w \in V$ such that $(w, v) \in E_a$ there is some $w' \in V'$ such that $(w', v') \in E'_a$ and $(w, w') \in B$.
back backwards: For all $a \in A$ and all $w' \in V'$ such that $(w', v') \in E_a$ there is some $w \in V$ such that $(w, v) \in E_a$ and $(w, w') \in B$.

For $v \in V$ and $v' \in V'$ we say that $(\mathfrak{A}, v)$ and $(\mathfrak{A}', v')$ are *bidirectional bisimilar* if there is a bidirectional bisimulation $B \subseteq V \times V'$ such that $(v, v') \in B$.

For a Kripke-structure $\mathfrak{A} = (V, (E_a)_{a \in A}, (P_i)_{i \in I})$ over the signature $\tau = \{E_a \mid a \in A\} \cup \{P_i \mid i \in I\}$ we define the Kripke-structure $\mathfrak{A}_B = (V, (E_a)_{a \in A}, (E_{a_B})_{a \in A}, (P_i)_{i \in I})$ over the signature $\tau_B = \{E_a \mid a \in A\} \cup \{E_{a_B} \mid a \in A\} \cup \{P_i \mid i \in I\}$ by $E_{a_B} = \{(v, u) \in V \times V \mid (u, v) \in E_a\}$ for $a \in A$.

Proposition 4.4. *Let* $\tau = \{E_a \mid a \in A\} \cup \{P_i \mid i \in I\}$ *and let* $\varphi \in (L_B)_\mu^k(\tau)$. *Furthermore let* $\mathfrak{A} = (V, (E_a)_{a \in A}, (P_i)_{i \in I})$ *and* $\mathfrak{A}' = (V', (E'_a)_{a \in A}, (P'_i)_{i \in I})$ *be Kripke-structures over the signature* τ. *Then the following propositions hold.*

(1) For all $\overline{v} \in V^k$ *and each valuation* $\mathcal{I}$ *of the variables into* V, *we have* $(\mathfrak{A}, \mathcal{I}, \overline{v}) \models \varphi$ *if and only if* $(\mathfrak{A}_B, \mathcal{I}, \overline{v}) \models \varphi \in L_\mu^k(\tau \cup \{E_{a_B} \mid a \in A\})$.

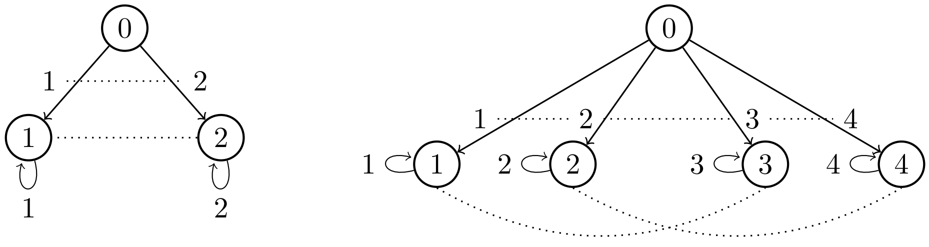

Figure 4.3: The games $\mathcal{G}$ and $\mathcal{H}$.

(2) $B \subseteq V \times V'$ is a bidirectional bisimulation between $\mathfrak{A}$ and $\mathfrak{A}'$ if and only if B is a bisimulation between $\mathfrak{A}_B$ and $\mathfrak{A}'_B$.

Clearly this yields, that the bidirectional multidimensional μ-calculus is invariant under bidirectional bisimulation. Again we can use this fact to derive the nondefinability of $\sim_1^*$ and $\sim_1^+$.

Corollary 4.7. *Let $\varphi \in (L_B)^k_\mu$ be a closed formula and let $\mathfrak{A}$, $\mathfrak{A}'$ be Kripke-structures. If $\overline{a} \in A^k$ and $\overline{a}' \in (A')^k$, such that $(\mathfrak{A}, a_j)$ and $(\mathfrak{A}', a'_j)$ are bidirectional bisimilar for $j = 1, \dots, k$, then $(\mathfrak{A}, \overline{a}) \models \varphi$ if and only if $(\mathfrak{A}', \overline{a}') \models \varphi$.*

Corollary 4.8. *Let $\# \in \{*, +\}$. Then there is no closed formula $\varphi \in (L_B)^2_\mu(\tau^\#)$, such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ we have $(\mathfrak{A}^\#_{\mathcal{G},v_0}, (\pi, \pi')) \models \varphi$ if and only if $\pi \sim_1^\# \pi'$.*

Proof. We define the games $G_1 = (V^1, V^1, (f_i)_{i=1,2})$ and $G_2 = (V^2, V^2, (f_i)_{i=1,\dots,4})$ by $V^1 = \{0, 1, 2\}$, $f_i(0) = f_i(i) = i$ for $i = 1, 2$, $V^2 = \{0, 1, 2, 3, 4\}$ and $f_i(0) = f_i(i) = i$ for $i = 1, \dots, 4$. Now consider the corresponding games $\mathcal{G}$ and $\mathcal{H}$ with partial information where $\mathrm{vis}_1^A(i) = 0$ for all actions i in both games. Furthermore $\mathrm{vis}_1^V(0) = 0$, $\mathrm{vis}_1^V(1) = \mathrm{vis}_1^V(2) = 1$ in $\mathcal{G}$ and $\mathrm{vis}_1^V(0) = 0$, $\mathrm{vis}_1^V(1) = \mathrm{vis}_1^V(3) = 1$, $\mathrm{vis}_1^V(2) = \mathrm{vis}_1^V(4) = 2$ in $\mathcal{H}$. The games are depicted in Figure 4.3.

Now let $\pi := 011$ and $\pi' := 022$. It is easy to see that $(\mathfrak{A}^\#_{\mathcal{G},0}, \pi)$ and $(\mathfrak{A}^\#_{\mathcal{H},0}, \pi)$ as well as $(\mathfrak{A}^\#_{\mathcal{G},0}, \pi')$ and $(\mathfrak{A}^\#_{\mathcal{H},0}, \pi')$ are bidirectional bisimilar. But $\mathrm{vis}_1^\#(\pi) = 001 = \mathrm{vis}_1^\#(\pi')$ in $\mathcal{G}$ and $\mathrm{vis}_1^\#(\pi') = 001 \neq 002 = \mathrm{vis}_1^\#(\pi')$ in $\mathcal{H}$. So using the bidirectional bisimulation invariance of L^2_μ, the proof is finished. ☐

Of course it is a crucial point that we require the formula φ to be closed. For consider the formula

$$\varphi := \mu \sim_1^* (([E_B]_1 0 \wedge [E_B]_2 0) \vee (\sim x_1 x_2 \wedge \langle E_B \rangle_1 \langle E_B \rangle_2 \sim_1^* x_1 x_2))$$

with the free relation symbol $\sim \in \mathrm{SV}_2$. Then for each game $\mathcal{G}$ with partial information, each position $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ we have

$$(\mathfrak{A}_{\mathcal{G},v_0}, (\sim \mapsto \sim_1), (\pi, \pi')) \models \varphi \iff (\pi, \pi') \in \sim_1^*.$$

But clearly this is not what we wanted to have. To conclude the considerations on the two-dimensional μ-calculus, we show that for each $n < \omega$ we can define the relation $\sim_1^*$ over the signature $\tau^n = \{E, P_1, \ldots, P_n\}$ if we restrict our attention to those games with partial information where we have at most n different $\sim_1$-equivalence classes. So we fix some $n < \omega$. Now let $\mathcal{G}$ be a game with partial information and let $v_0 \in P_{\text{fin}}(v_0)$ such that $|P_{\text{fin}}(v_0)/{\sim_1}| \leq n$. We define the τ^n-structure $\mathfrak{A}^n_{\mathcal{G},v_0} = (P_{\text{fin}}(v_0), E', P_1, \ldots, P_n)$ with E' as usual, $P_i = [\pi_i]_{\sim_1}$ for $i = 1, \ldots, m$ where $\{\pi_1, \ldots, \pi_m\}$ is a representative system for $P_{\text{fin}}(v_0)/{\sim_1}$ for some $m \leq n$ and $P_i = \emptyset$ for $i = m+1, \ldots, n$. Now we define

$$\varphi^n := \mu \sim_1^* (([E_B]_1 0 \wedge [E_B]_2 0) \vee (\bigwedge_{i=1}^{n} (P_i x_1 \leftrightarrow P_i x_2) \wedge \langle E_B\rangle_1 \langle E_B\rangle_2 \sim_1^* x_1 x_2)).$$

Then for all $\pi, \pi' \in P_{\text{fin}}(v_0)$ we have

$$(\mathfrak{A}^n_{\mathcal{G},v_0}, (\pi, \pi')) \models \varphi^n \iff \pi \sim_1^* \pi'.$$

Remark. Both constructions can also be carried out for the relation $\sim_1^+$ in a very similar way. Of course we would then have to add modalities $\langle E_0\rangle_i$ and $[E_0]_i$ which access the edge relation E_0' containing all the private moves of player 0.

4.3 Second Order Definability

4.3.1 Second Order Logic

First we show that each LFP-formula can be translated into an equivalent SO-formula and so from the fact that $\sim_1^*$ and $\sim_1^+$ are definable in LFP it follows that the relations are definable in SO as well.

Proposition 4.5. *Let τ be a signature. For each formula $\varphi(\overline{x}) \in \text{LFP}(\tau)$ we can effectively construct a formula $\varphi^*(\overline{x}) \in \text{SO}(\tau)$ such that for each τ-structure $\mathfrak{A}$ and all $\overline{a} \subseteq A$ we have $\mathfrak{A} \models \varphi(\overline{a})$ if and only if $\mathfrak{A} \models \varphi^*(\overline{a})$.*

Proof. We give an inductive translation $\varphi \mapsto \varphi^*$ which can easily be seen to be correct and which is obviously effective.

- $\varphi \mapsto \varphi$, if φ is an atomic formula.
- $\varphi_1 \wedge \varphi_2 \mapsto \varphi_1^* \wedge \varphi_2^*$.
- $\neg\psi \mapsto \neg\psi^*$.
- $\exists x\psi \mapsto \exists x\psi^*$
- $[\text{lfp } R\overline{x}\,\psi](\overline{t}) \mapsto \forall R\left[\forall\overline{x}(R\overline{x} \leftrightarrow \psi(\overline{x})) \rightarrow R\overline{t}\right]$.

□

4.3.2 Guarded Second Order Logic

Guarded logics have first been considered in the context of modal logics. The starting point was the question for the reasons of the good algorithmic properties of modal logics, especially the fact that the satisfiability problem for modal logics remains decidable if one adds advanced mechanisms like least and greatest fixed points (as we have done) or path quantification (which results in the computation tree logic CTL), see for example [Grä01]. (Notice that this is only true for plain modal logic and not for the multidimensional modal logic ML^k if $k \geq 2$. For example, the logic L_μ^2 does no longer have the finite model property (which L_μ in fact does have) and its satisfiability problem is undecidable, see [Ott99].) Andréka, van Benthem and Németi have suggested that the nice properties of modal logics are due to the fact that quantification in first order formulas which result from the translation of modal formulas is always *guarded*, that means, there are atomic formulas attached to the variables over which the quantifiers range. For a detailed discussion of this interesting issue, a formal definition of guarded first order and guarded fixed point logics and the references to the original work of Andréka, van Benthem and Németi and many other see [Grä01]. Now we introduce guarded second order logic GSO along the same lines as in [Hir02].

Definition 4.5. Let τ be a signature and let $\mathfrak{A}$ be a τ-structure.

(1) A finite set $\{a_1, \dots, a_k\} \subseteq A$ is called *guarded* in $\mathfrak{A}$ if there is an atomic formula $\alpha(x_1, \dots, x_k) \in \mathrm{FO}(\tau)$ where each x_i, $i = 1, \dots, n$ actually occurs in α, such that $\mathfrak{A} \models \alpha(a_1, \dots, a_k)$.
(2) A tuple $(a_1, \dots, a_k) \in A^k$ is called guarded in $\mathfrak{A}$ if $\{a_1, \dots, a_k\} \subseteq X$ for a guarded set $X \subseteq A$.
(3) A relation $R \subseteq A^k$ is called guarded in $\mathfrak{A}$ if each tuple $\overline{a} \in R$ is guarded in $\mathfrak{A}$.

Notice that we have to distinguish guardedness of a subset $M \subseteq A$, regarded as a set from guardedness of M, regarded as a monadic relation. (Which is clearly not the same according to the previous definition. A monadic relation is always guarded, since each tuple $\overline{a} = a \in A$ of length one is guarded by the atomic formula $\alpha(x) := x = x$ whereas a set of course is not always guarded.)

Syntax of GSO**.** The Syntax of GSO coincides with the syntax of SO.

Semantics of GSO**.** We consider only the case of quantification over second order variables. In all other cases, the semantics of GSO coincides with that of SO.

- $\mathfrak{A}, \mathcal{I}, \mathcal{J} \models_G \exists R\psi$ if and only if there is a *guarded* relation $\overline{R} \subseteq A^k$ such that $\mathfrak{A}, \mathcal{I}, \mathcal{J}[R \mapsto \overline{R}] \models \psi$.
- $\mathfrak{A}, \mathcal{I}, \mathcal{J} \models_G \forall R\psi$ if and only if for all *guarded* relations $\overline{R} \subseteq A^k$ we have $\mathfrak{A}, \mathcal{I}, \mathcal{J}[R \mapsto \overline{R}] \models \psi$.

The subscript G which is attached to the symbol for the modelship relation makes the difference between the two semantics explicit. This is useful since syntactically GSO and SO coincide. Nevertheless, if it is clear from the context which modelship relation we use, then we omit the index.

To answer the question whether we can define the relation $\sim_1^*$ in GSO we just have to look at the translation of the respective fixed point formulas into SO. So let ψ^* and φ_f^* be defined as in Section 4.2.1, that means, ψ^* defines $\sim_1^*$ as a least fixed point. The translation of ψ^* into SO is given by

$$\vartheta^*(x,y) := \forall R \left[\forall a \forall b (Rab \leftrightarrow \varphi_f^*(a,b,R)) \rightarrow Rxy \right].$$

Proposition 4.6. *For each game $\mathcal{G}$ with partial information, each $v_0 \in V$ and all $\pi, \pi' \in P_{\text{fin}}(v_0)$ we have $\mathfrak{A}_{\mathcal{G},v_0}^* \models_G \vartheta^*(\pi,\pi')$ if and only if $\pi \sim_1^* \pi'$.*

Proof. Let $\mathcal{G}$ be a game with partial information, let $v_0 \in V$ and let $\overline{R} \subseteq P_{\text{fin}}(v_0) \times P_{\text{fin}}(v_0)$. Then $\mathfrak{A}_{\mathcal{G},v_0}^* \models \forall a \forall b(\overline{R}ab \leftrightarrow \varphi_f^*(a,b,\overline{R}))$ if and only if $\overline{R}$ is a fixed point of the operator F^* as we have defined it in Section 4.2.1. Furthermore, $\text{lfp}(F^*) = \sim_1^* \subseteq \sim_1$ is a guarded relation and thus, if $\pi, \pi' \in P_{\text{fin}}(v_0)$ with $\mathfrak{A}_{\mathcal{G},v_0}^* \models_G \vartheta^*(\pi,\pi')$, then in particular we have $(\pi,\pi') \in \text{lfp}(F^*) = \sim_1^*$. Conversely, if $(\pi,\pi') \in \sim_1^* = \text{lfp}(F^*)$, then $(\pi,\pi') \in \overline{R}$ for any fixed point $\overline{R}$ of F^* and thus $\mathfrak{A}_{\mathcal{G},v_0}^* \models_G \vartheta^*(\pi,\pi')$. ☐

Now for the relation $\sim_1^+$ this argumentation cannot be applied, since $\sim_1^+$ is not guarded in general over the signature $\tau^+ = \{\sim, E, E_0\}$. One possibility to overcome this problem is to split the relation $\sim_1$ into the original relations $\sim_1^V$ and $\sim_1^A$ with $\pi \sim_1^V \pi'$ if and only if $\text{last}(\pi) \sim_1^V \text{last}(\pi')$ and $\pi a v \sim_1^A \pi' a' v'$ if and only if $a \sim_1^A a'$. Clearly the formula φ_f^+ (from Section 4.2.1) can be transformed into an equivalent formula over the signature $\sigma^+ = \{\sim^V, \sim^A, E, E_0\}$ by replacing the atomic formula $a \sim b$ in φ_f^+ with the formula $a \sim^V b \wedge a \sim^A b$. Furthermore, the relation $\sim_1^+$ is guarded over the signature σ^+ since $\pi \sim_1^+ \pi'$ entails $\text{last}(\pi) \sim_1^V \text{last}(\pi')$, that means, $\sim_1^+ \subseteq \sim_1^V$. So over the signature σ^+ we can apply the same argumentation as for $\sim_1^*$ to show that $\sim_1^+$ is GSO-definable.

Remark. As one can see, the non-definability results from this chapter also hold if we consider the signature σ^+ instead of τ^+. Furthermore, the definability results other than the result for GSO hold if we consider the signature σ^+ instead of τ^+ as well, since $x \sim y \equiv x \sim^V y \wedge x \sim^A y$. So in the case of GSO it is at least helpful to consider the signature σ^+ instead of τ^+; in all the other cases it doesn't matter.

4.3.3 Monadic Second Order Logic

The (plain) monadic second order logic (MSO) is obtained from second order logic by disallowing quantification over relations of arity at least 2. So MSO is a syntactic fragment of SO and the semantics of MSO is the same as for SO.

First we consider a variant of MSO, the so called MSO_2 logic which was motivated by questions concerning the expressiveness of MSO on graphs and where quantification does not only range over vertices but also over edges. Courcelle has studied this logic and its relation to usual MSO on graphs, for example in [Cou97].

So we consider the signature $\tau = \{E\}$. For a graph $G = (V,E)$ we define $\overline{G} := (V \uplus E, \overline{E})$ where $\overline{E} \subseteq V \times V \times E$ with $(u,v,e) \in \overline{E}$ if and only if e is an edge from u to v. Syntactically we have $\text{MSO}_2(\tau) := \text{MSO}(\{\overline{E}\})$. The semantics of $\text{MSO}_2(\tau)$ is defined via $G, \overline{v} \models_{\text{MSO}_2} \varphi(\overline{x})$ if and only if $\overline{G}, \overline{v} \models_{\text{MSO}} \varphi(\overline{x})$.

Of course this setting is not appropriate for our concerns, since we are dealing with at least two different edge relations which have to be distinguished. So we consider the signature $\sigma_n = \{E_1, \dots, E_n\}$ for some $n < \omega$ where E_i is a binary relation for each $i \in \{1, \dots, n\}$. For a graph $G = (V, E_1, \dots, E_n)$ we define $\overline{G} = (V \uplus E_1 \uplus \dots \uplus E_n, \overline{E_1}, \dots, \overline{E_n})$ where $\overline{E_i} \subseteq V \times V \times E_i$ with $(u, v, e) \in \overline{E_i}$ if and only if e is an i-edge from u to v for $i = 1, \dots, n$. Syntactically we have $\mathrm{MSO}_2(\sigma_n) = \mathrm{MSO}(\{\overline{E_1}, \dots, \overline{E_n}\})$. The semantics of $\mathrm{MSO}_2(\sigma_n)$ is defined via $G, \overline{v} \models_{\mathrm{MSO}_2} \varphi(\overline{x})$ if and only if $\overline{G}, \overline{v} \models_{\mathrm{MSO}} \varphi(\overline{x})$.

The following result shows that the MSO_2-definability of $\sim_1^*$ and $\sim_1^+$ follows from the GSO-definability of the two relations. This result can also be extended to arbitrary relational signatures, where $\mathrm{MSO}_2(\tau)$ for arbitrary relational signatures τ is defined in the obvious way. Furthermore, for each formula $\varphi \in \mathrm{MSO}_2(\tau)$ for a relational signature τ we can effectively construct an equivalent formula $\varphi^* \in \mathrm{GSO}(\tau)$. So GSO and MSO_2 are expressively equivalent over relational signatures. Since we do not need those results here we do not go into details.

Proposition 4.7. *For every formula* $\varphi(\overline{x}) \in \mathrm{GSO}(\{E_1, \dots, E_n\})$ *we can effectively construct a formula* $\varphi^*(\overline{x}) \in \mathrm{MSO}_2(\{E_1, \dots, E_n\})$ *such that for each graph* $G = (V, E_1, \dots, E_n)$ *and all* $\overline{v} \subseteq V$ *we have* $G \models_G \varphi(\overline{v})$ *if and only if* $G \models_{\mathrm{MSO}_2} \varphi^*(\overline{v})$.

Finally, we show that in plain monadic second order logic, neither $\sim_1^*$ nor $\sim_1^+$ are definable. Of course this immediately implies that the relations are not FO-definable either. But the method of Ehrenfeucht and Fraïssé which we have used to show that the relations are not FO-definable is quite important and offers a different and interesting perspective on the reasons for the non-definability. So both proofs are worth having a look at them.

First we show that $\sim_1^*$ is not MSO-definable and then we will briefly discuss how the arguments can be applied to show the non-definability of $\sim_1^+$ as well. The main argument for the proof is the theorem of Büchi, Elgot and Trakhtenbrot which says that the regular languages are exactly the MSO-definable languages, where a language $L \subseteq \Sigma^*$ for some alphabet Σ is called regular, if there is a finite automaton $\mathcal{A}$ such that $L(\mathcal{A}) = L$. For the original papers by Büchi, Elgot and Trakhtenbrot see for example the references in [Tho99]. Now we want to see how such languages can be defined in monadic second order logic.

Let Σ be an alphabet and let $\tau_\Sigma = \{S, \min, \max, <, (w_a)_{a \in \Sigma}\}$. For a word $w \in \Sigma^*$ we define the τ_Σ-structure $\underline{w} = (\mathrm{dom}(w), S, \min, \max, <, (w_a)_{a \in \Sigma})$ as follows.

- $\mathrm{dom}(w) = \{1, \dots, |w|\}$.
- $S(i) = i + 1$ for $i \in \mathrm{dom}(w) \setminus \{|w|\}$ and $S(|w|) = |w|$.
- min, max and $<$ as usual.
- $w_a = \{i \in \mathrm{dom}(w) \mid w_i = a\}$ for $a \in \Sigma$.

Now for an $\mathrm{MSO}(\tau_\sigma)$-sentence φ we define $L(\varphi) = \{w \in \Sigma^* \mid \underline{w} \models \varphi\}$ and we say that a language $L \subseteq \Sigma^*$ is MSO-definable, if there is an $\mathrm{MSO}(\tau_\Sigma)$ sentence φ such that $L = L(\varphi)$.

Theorem 4.6. *(Büchi, Elgot, Trakhtenbrot)*
A language $L \subseteq \Sigma^*$ *is regular if and only if it is* MSO*-definable.*

Proposition 4.8. *The language* $L = \{a^{n+1}b^n \mid 0 < n < \omega\} \subseteq \{a,b\}^*$ *is not regular.*

Proof. Assume that there is a deterministic finite automaton $\mathcal{A}$ which recognizes L. Then there is a natural number $n > 0$ such that, running on $a^{n+1}b^n$, $\mathcal{A}$ visits some state twice before the first b is seen. Thus, there is a natural number $k > 0$ such that $\mathcal{A}$ accepts the word $a^{n+1+k}b^n$ in contradiction to $L(\mathcal{A}) = L$. □

Theorem 4.7. *The class* $\mathcal{K}_1^*$ *is not* MSO*-definable in* $\mathcal{K}^*$*.*

Proof. Assume that there is a sentence $\varphi \in \mathrm{MSO}(\tau^*)$ such that for all $\mathfrak{A} \in \mathcal{K}^*$ we have $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \in \mathcal{K}_1^*$. We show that from this we can infer that the language $\{a^{n+1}b^n \mid 0 < n < \omega\}$ is regular. For this purpose, for a word $w \in \{a,b\}^*$, we define the τ^*-structure $\mathfrak{A}_w = (\mathrm{dom}(w), E^w, \sim^w)$ as follows.

- $\sim^w = \mathrm{dom}(w) \times \mathrm{dom}(w) \setminus \{(i,j),(j,i) \mid \; i \neq j \wedge w_i = a \wedge w_{i+1} = b \;\}$.
- E^w is the union of the following sets.
  - $\{(i, i-1) \mid i \in \mathrm{dom}(w) \setminus \{1\}, w_i = a\}$.
  - $\{(i, i+1) \mid i \in \mathrm{dom}(w) \setminus \{|w|\}, w_i = b\}$.
  - $\{(i, i+1) \mid i \in \mathrm{dom}(w) \setminus \{|w|\}, w_i = a, w_{i+1} = b\}$.

Now we define the following formulas α, $\varphi_\sim$ and φ_E over the signature $\tau_{\{a,b\}}$.

- $\alpha := \exists x((\min < x < \max) \wedge \forall y((y \leq x \rightarrow w_a y) \wedge (x < y \rightarrow w_b y)))$.
- $\varphi_\sim(x,y) := \neg(x \neq y \wedge w_a x \wedge w_b Sx) \wedge \neg(x \neq y \wedge w_a y \wedge w_b Sy)$.
- $\varphi_E(x,y)$ is the disjunction over the following formulas.
  - $\min < x \wedge w_a x \wedge Sy = x$.
  - $x < \max \wedge w_b x \wedge Sx = y$.
  - $x < \max \wedge w_a x \wedge w_b y \wedge Sx = y$.

Now let $\psi = \alpha \wedge \varphi(\sim/\varphi_\sim, E/\varphi_E)$, where $\varphi(\sim/\varphi_\sim, E/\varphi_E)$ is obtained from φ by replacing each occurrence of $\sim$ and E with $\varphi_\sim$ and φ_E, respectively. By an easy induction over the structure of the formulas we can show that for each $w \in \{a,b\}^*$ and all $\eta \in \mathrm{MSO}(\tau^*)$ we have $\underline{w}, \mathcal{I} \models \eta(\sim/\varphi_\sim, E/\varphi_E)$ if and only if $\mathfrak{A}_w, \mathcal{I} \models \eta$, where $\mathcal{I}$ is an arbitrary interpretation of the free (first and second order) variables in η.

Finally, we show that for all $w \in \{a,b\}^*$ we have $\underline{w} \models \psi$ if and only if $w = a^{n+1}b^n$ for some $n \in \mathcal{N} \setminus \{0\}$. Then according to the theorem of Büchi, Elgot and Trakhtenbrot we can conclude that $\{a^{n+1}b^n \mid 0 < n < \omega\}$ is a regular language which yields the desired contradiction.

So let $w \in \{a,b\}^*$ and let first $\underline{w} \models \psi$. Then we have $\underline{w} \models \alpha$ and so, $w = a^k b^m$ for some $k, m < \omega$ with $m \geq 1$ and $k \geq 2$. Furthermore, $\underline{w} \models \varphi(\sim/\varphi_\sim, E/\varphi_E)$, so

$\mathfrak{A}_w \models \varphi$. Using $w = a^k b^m$ it is easy to see that $\mathfrak{A}_w \in \mathcal{K}^*$, and thus, $\mathfrak{A}_w \models \varphi$ yields $\mathfrak{A}_w \in \mathcal{K}_1$. But from this we can obviously infer that $k-1 = m$, so $a^k b^m = a^{m+1} b^m$.

Now let conversely $w = a^{n+1} b^n$ for some $0 < n < \omega$. Obviously $\underline{w} \models \alpha$. Furthermore it is easy to see that $\mathfrak{A}_w \in \mathcal{K}_1^*$ and so we have $\mathfrak{A}_w \models \varphi$. This yields $\underline{w} \models \varphi(\sim/\varphi_\sim, E/\varphi_E)$, and so we have $\underline{w} \models \psi$. $\square$

Corollary 4.9. *There is no formula $\varphi(x,y) \in \mathrm{MSO}(\tau^*)$ such that for all games $\mathcal{G}$ with partial information, all $v_0 \in V$ and all $\pi, \pi' \in P_{\mathrm{fin}}(v_0)$ we have $\mathfrak{A}^*_{\mathcal{G},v_0} \models \varphi(\pi,\pi')$ if and only if $\pi \sim_1^* \pi'$.*

The proof of this corollary is completely analog to the proof of Corollary 4.2.

Now we want to see how we can extend this solution to show that $\sim_1^+$ is not MSO-definable. We do not provide a full proof since the techniques are exactly the same as in the proof of Theorem 4.7. For a word $w \in \{a,b\}^*$ let the τ^+-structure $\mathfrak{A}_w = (\mathrm{dom}(w), E_0^w, E^w, \sim^w)$ be defined as follows.

- $E_0^w = \emptyset$.
- E^w is the edge relation from the proof of Theorem 4.7.
- $\sim^w = \mathrm{dom}(w) \times \mathrm{dom}(w) \setminus ((X_1 \cup X_2 \cup X_3) \setminus X_4)$ with
  - $X_1 = \{(i,j),(j,i) \mid i \neq j \wedge w_i = a \wedge w_{i+1} = b\}$,
  - $X_2 = \{(i+1,j),(j,i+1) \mid i+1 \neq j \wedge w_i = a \wedge w_{i+1} = b\}$,
  - $X_3 = \{(i-1,j),(j,i-1) \mid i-1 \neq j \wedge w_i = a \wedge w_{i+1} = b\}$,
  - $X_4 = \{(i-1,i+1),(i+1,i-1) \mid w_i = a \wedge w_{i+1} = b\}$.

Now let α and φ_E be as in the proof of Theorem 4.7. We define the formulas $\varphi_{E_0}(x,y)$ and $\varphi_\sim(x,y)$ as follows.

- $\varphi_{E_0}(x,y) := x = y \wedge x \neq y$.
- $\varphi_\sim(x,y) :=$

$$\begin{aligned}
&\neg(x \neq y \wedge w_a x \wedge w_b Sx) \wedge \neg(x \neq y \wedge w_a y \wedge w_b Sy)\\
&\wedge \neg(x \neq y \wedge \exists z(Sz = x \wedge w_a z \wedge w_b x)) \wedge \neg(x \neq y \wedge \exists z(Sz = y \wedge w_a z \wedge w_b y))\\
&\wedge \neg(x \neq y \wedge w_a Sx \wedge w_b SSx) \wedge \neg(x \neq y \wedge w_a Sy \wedge w_b SSy)\\
&\vee (SSx = y \wedge w_a S_x \wedge w_b y) \vee (SSy = x \wedge w_a S_y \wedge w_b x)
\end{aligned}$$

Now assume that there is some $\psi \in \mathrm{MSO}(\tau^+)$ such that ψ defines the class $\mathcal{K}_1^+$ in the class $\mathcal{K}^+$, where $\mathcal{K}_1^+$ is defined as in Section 4.1. As in the proof of Theorem 4.7 we can show that $L(\alpha \wedge \psi(E_0/\varphi_{E_0}, E/\varphi_E, \sim /\varphi_\sim)) = \{a^{n+1}b^n \mid 0 < n < \omega\}$ which yields a contradiction to the fact that the latter language is not regular.

Remark. Notice that in both cases we could use the same trick as for the first order case to prove the previous results under the assumption that there are no terminal positions in games. Though, since there are selfloops on the additional positions, the unravelling of the game graphs will be infinite. So we would have to deal with ω-languages and use the ω-version of the theorem of Büchi, Elgot and Trakhtenbrot, cf. Section 3.1.3. Notice that the language $\{a^{n+1}b^nc^\omega \mid 0 < n < \omega \} \subseteq \{a,b,c\}^\omega$ is not ω-regular.

Chapter 5

Conclusion and Future Work

In this thesis, we have introduced a general model for two-player games with partial information and we have considered two different possibilities to define the local state of a player after some finite play prefix has been played. We have solved the strategy problem for both resulting notions of strategies up to arbitrary ω-regular winning conditions and we have discussed the connection between the nonemptiness problem for alternating tree automata and the strategy problem for ω-regular games with partial information. Upper and lower bounds for the memory which is needed to implement winning strategies in certain classes of games with partial information have been established. Furthermore, the logical definability of the equivalence relations on finite play prefixes which result from the two definitions of local states has been analyzed.

Altogether, this work makes some contributions to problems in the context of games with partial information which have not been considered very much before and it likes to open some new perspectives for the theory. Many of the results in this thesis can and should be sharpened or extended in a certain way. For example the exact complexity of the strategy problem for the case of arbitrary omega-regular winning conditions has not been considered here. Furthermore, the conjecture from Section 3.3 should be proved and it would be interesting to have a look at the complexity of the alternative method for solving the nonemptiness problem for tree automata which has been suggested in [KV05] and [KPV06], if it is applied directly to universal automata.

It would also be interesting to figure out certain “natural” properties for the partial information in games which yield a lower complexity for the strategy problem. Examples are information compatibility, blindfolding player 1 and assuming that $\sim_1^A$ implies $\sim_1^V$. If we put all these properties together, then the strategy problem for parity games with partial information is PSPACE-complete. But if we don’t assume that player 1 is blindfolded, then the strategy problem for information compatible parity games is again EXPTIME-complete.

Of course we could also consider stronger conditions, like for example bounded size of equivalence classes of positions. We have already noticed that the strategy problem for a class of finite information compatible Büchi-games with bounded size of equivalence classes of positions can be solved in polynomial time. Furthermore, it surely has some impact on certain measures for the graph complexity of the game graph of the corresponding game with full information, like tree width, DAG-width and entanglement. The only examples of classes of games with partial information and bounded tree width which have been found during this work, such that there is no bound on the tree width of the corresponding games with full information, require unbounded size of equivalence classes of positions.

Another possibility to prove a lower complexity for the strategy problem for certain classes of games with partial information would be to figure out general conditions, under which the method for the evaluation of μ-calculus formulas from Section 3.7 works faster than exponential time. Of course, this depends highly on the implementation of the involved operators, especially the controllable predecessor operator $\mathrm{CPre}_1^{\mathcal{A}}$. In [dWDHR06], the method has been used to solve the universality problem for finite nondeterministic automata (NFA). There, a class of NFA has been suggested for which the resulting algorithm is exponentially faster than the "classical" algorithm. (Notice that in this case, the corresponding game with full information is deterministic and player 0 has no real influence on the game. So the controllable predecessor operator $\mathrm{CPre}_1^{\mathcal{A}}$ reduces to a much simpler form.)

Further Aspects. The two aspects of games with partial information that we have mainly considered in this work, are the existence of winning strategies in games with two players and the logical definability of the local states of the players. In Section 2.2 we have discussed certain possibilities to get a finite representation of local states. Of course these methods should be investigated much further. Moreover, in the introduction we have already mentioned multiplayer games and model checking games for independent logics which offer a lot of potentialities. We have also seen that essentially, the strategy problem coincides with the problem whether player 1 has a strategy which is winning against all strategies of player 0. So we should look for other questions, where the partial information of all players is involved at the same time de facto.

Equilibria. There are certain notions of equilibria for (multiplayer, non-zero sum) games. The most well known examples are Nash-equilibria and subgame perfect equilibria. There are also certain special notions of equilibria for games with partial information. For example, for extensive form games with imperfect information, the notion of sequential equilibria exists and for Bayes-games the Nash-equilibria of a corresponding strategic form game are used as equilibria. We can ask, for example, under which assumptions we can guarantee the existence of a certain equilibrium in a (multiplayer, non-zero sum) game with partial information. Or we can ask for the complexity of computing such an equilibrium.

Cooperation. Cooperation means that certain players play together to achieve a common goal, independently of how all the other players play (of course there might be no others players left at all). So we ask the following question. Given a set W of plays from a position v_0 and a set $N' \subseteq N$ of players, is there a set $\mathfrak{F}$ of strategies, one for each player from N', such that each play from initial position v_0 that is compatible with all those strategies, belongs to W? (Notice that for appropriate W, this question can be formalized in alternating time temporal logic, [AHK02].) Now the point is that each of the players from N' might have different knowledge, so this problem is very different from the problem of finding a winning strategy for one player in a game with partial information.

Dominated Strategies. A strategy f for a player $n \in N$ is called dominated, if there is a strategy g for player n, such that for all strategy profiles $(f_m)_{m \neq n}$, the outcome of $(f, f_m)_{m \neq n}$ is at most the outcome of $(g, f_m)_{m \neq n}$ and there is a strategy profile $(f_m)_{m \neq n}$ such that the outcome of $(f, f_m)_{m \neq n}$ is less than the outcome of $(g, f_m)_{m \neq n}$. Now we can ask, for example, given a set $\mathfrak{F}$ of strategies of player n, is there a dominated strategy in $\mathfrak{F}$? Of course we have to find some finite representation of such sets $\mathfrak{F}$ of strategies. (Like we have done for the set of all (winning) strategies in two-player games, using universal tree automata.)

Probability of Winning 1. Consider a game with nondeterministic actions, where for each action there is a probability distribution over the edges which are labelled with this action. (A special case are games where all the moves of each player are deterministic but there are some moves of “nature”, which are probabilistic.) Now if all players fix a strategy for such a game, then this does not yield a play of the game, but a probability distribution over plays. So we can for example ask for the value $\sup_\sigma \inf_\tau p(\sigma, \tau)$ where σ ranges over all partial information strategies for player 1, τ ranges over all partial information strategies of player 2 and $p(\sigma, \tau)$ is the probability that player 1 wins if the strategies σ and τ are chosen. We can also consider games which are not win-loss games and ask for the value $\sup_\sigma \inf_\tau e(\sigma, \tau)$, where $e(\sigma, \tau)$ is the expected value for player 1, if the strategies σ and τ are chosen and so on.

Probability of Winning 2. Another possibility is to give the players the possibility to randomize their strategies. In this setting we can again ask for the value $\sup_\sigma \inf_\tau p(\sigma, \tau)$ where σ ranges over all randomized partial information strategies for player 1, τ ranges over all randomized partial information strategies of player 2 and $p(\sigma, \tau)$ is the probability that player 1 wins if the strategies σ and τ are chosen. Or we can ask whether there is a randomized partial information strategy σ for player 1, such that for all randomized strategies τ of player 2 we have $p(\sigma, \tau) = 1$. Such a strategy is called an almost sure winning strategy. It should be noticed that for two-player zero-sum games with full information, deterministic strategies suffice to win, while this is not the case for games with partial information. A proof of this fact and an algorithm for computing the almost sure winning states for a player in an information compatible Büchi-game can be found in [CDHR06].

General Stochastic Games. Stochasticity is widely used in classical game theory, not only (and even not mainly) in the context of winning strategies. For example, Nash’s Theorem says that each strategic game (with finitely many actions for each player and von Neumann preferences over mixed strategy profiles) has a mixed-strategy Nash-equilibrium. Furthermore, Bayes-games, which are strategic games with incomplete information, use probabilities to model this incomplete information. We have already mentioned these games in the introduction. In [Sor], Sorain used a variant of this model, where only two-player zero-sum games are considered, but the game is repeated.

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Appendix A

A Note on $\mathrm{CPre}_1^{\mathcal{A}}$

We have mentioned that the efficiency of the method from Section 3.7 for the evaluation of μ-calculus formulas on games which result from the universal powerset construction depends highly on the implementation of the involved operators, especially the controllable predecessor operator $\mathrm{CPre}_1^{\mathcal{A}}$. So we shall have a short look at some plain implementations of this operator.

The first implementation which consists of Algorithm 1 and Algorithm 2 is a simple bottom up approach. That is, we start with singleton sets $\{x\}$ which belong to $\mathrm{CPre}_1^{\subseteq}(q)$ and then we put them together, until we have constructed all the elements from $\mathrm{CPre}_1^{\subseteq}(q)$. Afterwards we eliminate dominated elements. Clearly, this implementation makes heavy use of the fact that $\mathrm{CPre}_1^{\subseteq}$ is downward closed. However, it does not take advantage of the fact that we are dealing with maximal sets only. Indeed, if we skip the elimination of dominated sets, then the algorithm computes the operator CPre_1 on games which result from the universal powerset construction.

Algorithm 3 computes the set $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$ rather efficiently, but it does not work for the set $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$. The idea is to compute the sets $\mathrm{Pre}_a(T)$ of all a-predecessors of T for $T \in q$. If we partition these sets into equivalence classes, then the resulting sets can easily be seen to be in $\mathrm{CPre}_1^{\subseteq}(q)$. Then, again, we put those sets together until we have computed all sets which belong to $\mathrm{CPre}_1^{\subseteq}(q) \cap V_1^u$. The advantage is that we start with sets which are already much "closer" to the maximal sets in $\mathrm{CPre}_1^{\subseteq}(q)$ than singleton sets. Furthermore, in this case, very little information is needed to decide, whether two sets can be put together.

Finally, the third implementation uses a top down approach. That is, we start from sets $\mathrm{Pre}_A(\bigcup q) \cap V_0$, which are partitioned into equivalence classes. The resulting sets are of course not necessarily in $\mathrm{CPre}_1^{\mathcal{A}}(q)$, so we consider subsets of these sets by successively removing elements. As soon as we find a set, which belongs to $\mathrm{CPre}_1^{\subseteq}(q)$, this set belongs already to $\mathrm{CPre}_1^{\mathcal{A}}(q)$. So if the maximal elements in $\mathrm{CPre}_1^{\subseteq}(q)$ form reasonable "approximations" of the sets which we start with, then this implementation may be very efficient.

In the following, let $\mathcal{G} = (G, (\mathrm{vis}_i^V), (\mathrm{vis}_i^A))$ with $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$ be a finite information compatible parity game such that $\mathrm{act}([u]_{\sim_1}) \neq \emptyset$ for all $u \in V_1$. Furthermore let $G^u = (V^u, V_0^u, (E_a^u)_{a \in A^u}, \mathrm{col}^u)$ be the corresponding universal game with full information and let $\mathcal{A} = \{\lceil W \rceil \mid W \subseteq V^u\}$.

For the presentation of the algorithms we need some notation.

- For $B \subseteq A$ and $T \subseteq V$ let
  $\mathrm{Pre}_B(T) := \{w \in V \mid \; \exists a \in B \cap \mathrm{act}(w) : f_a(w) \in T \; \}$
- For $\sigma \in \mathrm{VIS}_1^V =: \{\sigma_1, \ldots, \sigma_r\}$ let $[\sigma] := (\mathrm{vis}_1^V)^{-1}(\{\sigma\})$.
- For $S, S' \subseteq V$ let $S \sim S'$ if $S \cup S' \subseteq [\sigma]$ for some $\sigma \in \mathrm{VIS}_1^V$.
- Let $\overline{A} =: \{a_1, \ldots, a_k\}$ be a representative system for $A/{\sim_1^A}$ with $A_1 \subseteq \overline{A}$.

Algorithm 1: Compute $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$

```
forall $a \in A_1$
    $X := \mathrm{Pre}_a(\bigcup q) \cap V_1$
    forall $(x, \sigma) \in X \times \mathrm{VIS}_1^V$
        if $f_a(x) \in [\sigma]$ : $T^\sigma(\{x\}) := \{f_a(x)\}$, else : $T^\sigma(\{x\}) := \emptyset$
    $\mathrm{CPre}_a := \{\{x\} \mid x \in X\}$
    forall $x \in X$
        $X' := \emptyset$
        forall $S \in \mathrm{CPre}_a$
            if $\{x\} \sim S$ and $\forall \sigma \in \mathrm{VIS}_1^V \; \exists U \in q : T^\sigma(\{x\}) \cup T^\sigma(S) \subseteq U$
                $S' := S \cup \{x\}$
                $X' := X' \cup \{S'\}$
                forall $\sigma \in \mathrm{VIS}_1^V$ : $T^\sigma(S') = T^\sigma(S) \cup T^\sigma(\{x\})$
            endif
        endfor
        $\mathrm{CPre}_a := \mathrm{CPre}_a \cup X'$
   endfor
endfor
$\mathrm{CPre}_1 := \bigcup\{\mathrm{CPre}_a \mid a \in A_1\}$
$Y := \emptyset$
while $(\mathrm{CPre}_1 \setminus Y \neq \emptyset)$
    choose $S \in \mathrm{CPre}_1 \setminus Y$, $Y := Y \cup \{S\}$
    forall $S' \in \mathrm{CPre}_1 \setminus Y$
        if $S' \subseteq S$ : $\mathrm{CPre}_1 := \mathrm{CPre}_1 \setminus \{S'\}$
        if $S \subseteq S'$ : $\mathrm{CPre}_1 := \mathrm{CPre}_1 \setminus \{S\}$
    endfor
endwhile
output $\mathrm{CPre}_1$
```

Algorithm 2: Compute $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$

```
$X := \mathrm{Pre}_A(\bigcup q) \cap V_0$
$X' := X$
while $X' \neq \emptyset$
    choose $x \in X'$, $X' := X' \setminus \{x\}$
    forall $(a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V$ : $T_a^\sigma(\{x\}) := \mathrm{Post}_{[a]_{\sim_1}}(\{x\}) \cap [\sigma]$
    if $\exists (a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V \; \forall U \in q : T_a^\sigma(\{x\}) \not\subseteq U$ : $X := X \setminus \{x\}$
endwhile
$\mathrm{CPre}_0 := \{\{x\} \mid x \in X\}$
```

```
forall $x \in X$
    $X' := \emptyset$
    forall $S \in \mathrm{CPre}_0$
        if $\{x\} \sim S$ and $\forall (a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V \; \exists U \in q : T_a^\sigma(\{x\}) \cup T_a^\sigma(S) \subseteq U$
            $S' := S \cup \{x\}$
            $X' := X' \cup \{S'\}$
            forall $(a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V : T_a^\sigma(S') = T_a^\sigma(S) \cup T_a^\sigma(\{x\})$
        endif
    endfor
    $\mathrm{CPre}_0 := \mathrm{CPre}_0 \cup X'$
endfor
$Y := \emptyset$
while $(\mathrm{CPre}_0 \setminus Y \neq \emptyset)$
    choose $S \in \mathrm{CPre}_0 \setminus Y$, $Y := Y \cup \{S\}$
    forall $S' \in \mathrm{CPre}_0 \setminus Y$
        if $S' \subseteq S : \mathrm{CPre}_0 := \mathrm{CPre}_0 \setminus \{S'\}$
        if $S \subseteq S' : \mathrm{CPre}_0 := \mathrm{CPre}_0 \setminus \{S\}$
    endfor
endwhile
output $\mathrm{CPre}_0$
```

Proposition A.1.
(1) $\mathrm{CPre}_1 = \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$.
(2) $\mathrm{CPre}_0 = \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$.

Proof. (1) Let first $S = \{x_1, \ldots, x_k\} \in \mathrm{CPre}_a$ for some $a \in A_1$. Then clearly $a \in \mathrm{act}(S)$, and for each $i = 1, \ldots, k$ there is some $U \in q$ such that $f_a(x_i) \in U$. Now the following condition is invariant under the for-loop over X. $S \subseteq [\tau]$ for some $\tau \in \mathrm{VIS}_1^V$ and $T^\sigma(S) = \mathrm{Post}_a(S) \cap [\sigma] \subseteq U$ for some $U \in q$ for all $\sigma \in \mathrm{VIS}_1^V$. This yields $S \in \mathrm{CPre}_1^{\subseteq}(q)$.

Now let $S = \{x_1, \ldots, x_k\} \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$ and let $a \in \mathrm{act}(S) \subseteq A_1$ such that for each $(S, T) \in E_a^u$ there is some $U \in q$ with $T \subseteq U$, that means, for each $\sigma \in \mathrm{VIS}_1^V$ there is some $U \in q$ such that $\mathrm{Post}_a(S) \cap [\sigma] \subseteq U$. Then clearly for each $i \in \{1, \ldots, k\}$ there is some $U \in q$ such that $f_a(x_i) \in U$. Thus we have $\{x_i\} \in \mathrm{CPre}_a$ for all $i \in \{1, \ldots, k\}$. If $k = 1$ this yields $S \in \mathrm{CPre}_a$, so let $k > 1$ and consider the first time some element from S, say x_1, is chosen in some pass of the for-loop over X in the first algorithm. Then obviously the set $\{x_1, x_2\}$ is put into CPre_a. If $k = 2$ we are done, so let $k > 2$ and consider the first time some element from $S \setminus \{x_1, x_2\}$, say x_3, is chosen in some pass of the for-loop over X. Then $\{x_1, x_2, x_3\}$ is put into CPre_a and so on until S is put into CPre_a.

Now using these two facts it can easily be shown that each $S \in \mathrm{CPre}_1$ is maximal in $\mathrm{CPre}_1^{\subseteq}(q)$ and that each $S \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$ is maximal in CPre_1, before dominated elements are eliminated.

(2) Let first $S = \{x_1, \ldots, x_k\} \in \mathrm{CPre}_0$. Then for each $(i, a, \sigma) \in \{1, \ldots, k\} \times \overline{A} \times \mathrm{VIS}_1^V$ there is some $U \in q$ such that $\mathrm{Post}_{[a]_{\sim_1}}(\{x_i\}) \cap [\sigma] \subseteq U$. Now the following condition is invariant under the for-loop over X. $S \subseteq [\tau]$ for some $\tau \in \mathrm{VIS}_1^V$ and $T_a^\sigma(S) = \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [\sigma] \subseteq U$ for some $U \in q$ for all $(a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V$. This yields $S \in \mathrm{CPre}_0^{\subseteq}(q)$.

Now let $S = \{x_1, \dots, x_k\} \in \mathrm{CPre}_0^{\mathcal{A}}(q) \cap V_0^u$. Then for each $i \in \{1, \dots, k\}$ there is some $a \in \mathrm{act}(x_i) \subseteq A_0$ and we have $a \in act(S)$. Since $\mathrm{Post}_{[a]_{\sim_1}}(\{x_i\}) \cap [f_a(x_i)]_{\sim_1} \subseteq U$ for some $U \in q$ we have $x_i \in \mathrm{Pre}_{A_0}(\bigcup q)$. Furthermore for each $(a, \sigma) \in \overline{A} \times \mathrm{VIS}_1^V$ there is some $U \in q$ such that $\mathrm{Post}^{\sigma}_{[a]_{\sim_1}}(\{x_i\}) \subseteq U$. So we have $\{x_i\} \in CPre_0$ before dominated elements are eliminated, for all $i \in \{1, \dots, k\}$. With the same arguments as in (1) we can now show that $S \in \mathrm{CPre}_0$, before dominated elements are eliminated.

Again, using these two facts it can easily be shown that each $S \in \mathrm{CPre}_0$ is maximal in $\mathrm{CPre}_1^{\subseteq}(q)$ and that each $S \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$ is maximal in CPre_0, before dominated elements are eliminated. ☐

Algorithm 3: Compute $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$

```
forall $a \in A_1$
    $X := \emptyset$
    forall $(T, \tau) \in q \times \mathrm{VIS}_1^V$
        $S := \mathrm{Pre}_a(T) \cap [\tau] \cap V_1$
        if $\forall S' \in X : S \not\subseteq S'$
            $X := X \cup \{S\}$
            forall $\sigma \in \mathrm{VIS}_1^V$ : if $T \subseteq [\sigma]$ : $\zeta(S, \sigma) := 1$, else $\zeta(S, \sigma) := 0$
        endif
    endfor
    $\mathrm{CPre}_a := X$
    forall $S \in X$
        $X' := \emptyset$
        forall $S' \in \mathrm{CPre}_a$
            if $S' \sim S$ and $\forall \sigma \in \mathrm{VIS}_1^V : \zeta(S, \sigma) = 0$ or $\zeta(S', \sigma) = 0$
                $S'' := S \cup S'$, $X' := X' \cup \{S''\}$
                forall $\sigma \in \mathrm{VIS}_1^V : \zeta(S'', \sigma) := \max\{\zeta(S, \sigma), \zeta(S', \sigma)\}$
            endif
        endfor
        $\mathrm{CPre}_a := \mathrm{CPre}_a \cup X'$
    endfor
endfor
$\mathrm{CPre}_1 := \bigcup\{\mathrm{CPre}_a \mid a \in A_1\}$
$Y := \emptyset$
while $(\mathrm{CPre}_1 \setminus Y \neq \emptyset)$
    choose $S \in \mathrm{CPre}_1 \setminus Y$, $Y := Y \cup \{S\}$
    forall $S' \in \mathrm{CPre}_1 \setminus Y$
        if $S' \subseteq S$ : $\mathrm{CPre}_1 := \mathrm{CPre}_1 \setminus \{S'\}$
        if $S \subseteq S'$ : $\mathrm{CPre}_1 := \mathrm{CPre}_1 \setminus \{S\}$
    endfor
endwhile
output $\mathrm{CPre}_1$
```

Proposition A.2. $\mathrm{CPre}_1 = \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$.

Proof. First let $S \in \mathrm{CPre}_a$ for some $a \in A_1$. Then there are $S_1, \dots, S_m \subseteq V_1$, $T_1, \dots, T_m \in q$ and $\tau_1, \dots, \tau_m$ such that $S_i = \mathrm{Pre}_a(T_i) \cap [\tau_i]$ for $i = 1, \dots, m$ and $S = \bigcup\{S_i \mid i = 1, \dots, m\}$. Thus, for $i = 1, \dots, m$ we have $\mathrm{Post}_a(S_i) \cap [\sigma] \subseteq \mathrm{Post}_a(S_i) \subseteq T_i$

for all $\sigma \in \mathrm{VIS}_1^V$. Now the following condition is invariant under the for-loop over X. $S \subseteq [\tau]$ for some $\tau \in \mathrm{VIS}_1^V$, $\zeta(S,\sigma) = 1$ if and only if $\mathrm{Post}_a(S) \cap [\sigma] \neq \emptyset$ and for each $\sigma \in \mathrm{VIS}_1^V$ there is at most one $i \in \{1,\ldots,m\}$ such that $\zeta(S_i,\sigma) = 1$. This yields $S \in \mathrm{CPre}_1^{\subseteq}(q) \cap V_1^u$.

Now let $S \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$ and let $a \in \mathrm{act}(S) \subseteq A_1$ such that for all $(S,T) \in E_a^u$ there is some $U \in q$ with $T \subseteq U$, that means, for all $\sigma \in \mathrm{VIS}_1^V$ there is some $U \in q$ such that $\mathrm{Post}_a(S) \cap [\sigma] \subseteq U$. Now let $\tau \in \mathrm{VIS}_1^V$ such that $S \subseteq [\tau]$ and let $SE_a^u = \{T_1,\ldots,T_m\}$. For $i \in \{1,\ldots,m\}$ we define $S_i := \mathrm{Pre}_a(T_i) \cap [\tau]$.

First we show that $S = \bigcup\{S_i \,|\, i = 1,\ldots,m\}$. If $v \in S$ then $a \in \mathrm{act}(v)$ and $f_a(v) \in T_i$ for some $i \in \{1,\ldots,m\}$ since $S \in \mathrm{CPre}_1^{\mathcal{A}}(q)$. Thus, $v \in S_i$. The converse inclusion follows from the maximality of S in $\mathrm{CPre}_1^{\subseteq}(q)$ because obviously $S \subseteq \bigcup\{S_i \,|\, i = 1,\ldots,m\} \in \mathrm{CPre}_1^{\subseteq}(q)$. Furthermore, with the same argument, there cannot be any $T \in q$ such that $S_i \subsetneq \mathrm{Pre}_a(T) \cap [\tau]$ for some $i \in \{1,\ldots,m\}$. ($S \subsetneq \bigcup\{S_j \,|\, j \neq i\} \cup \mathrm{Pre}_a(T) \cap [\sigma] \in \mathrm{CPre}_1^{\subseteq}(q)$.) Thus we have $S_i \in X$ for $i = 1,\ldots,m$.

Now for $i \in \{1,\ldots,m\}$ let $\sigma_i \in \mathrm{VIS}_1^V$ such that $T_i \subseteq [\sigma_i]$. Then clearly $\sigma_i \neq \sigma_j$ for $i,j \in \{1,\ldots,m\}$ with $i \neq j$ and so have $\zeta(S_i,\sigma_j) = 1$ if $i = j$ and $\zeta(S_i,\sigma_j) = 0$ if $i \neq j$. From these observations we can easily infer that $S \in \mathrm{CPre}_a$. (If $m = 1$ this is obvious. If $m > 1$, consider the first time some element from $\{S_i \,|\, i = 1,\ldots,m\}$, say S_1, is chosen in some pass of the for-loop over X and consider the first time some element from $\{S_i \,|\, i = 2,\ldots,m\}$, say S_2, is chosen in some pass of the for-loop over CPre_a. Then $S_1 \cup S_2$ is put into CPre_a and so on.)

Now using these two facts it can easily be shown that each $S \in \mathrm{CPre}_1$ is maximal in $\mathrm{CPre}_1^{\subseteq}(q)$ and that each $S \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$ is maximal in CPre_1, before dominated elements are eliminated. ☐

Algorithm 4: Compute $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$

```
$X := \emptyset$, $\mathrm{Pre} := \mathrm{Pre}_{A_1}(\bigcup q) \cap V_1$
forall $\tau \in \mathrm{VIS}_1^V$
    $S := \mathrm{Pre} \cap [\tau]$
    forall $(a,\sigma) \in A_1 \times \mathrm{VIS}_1^V$ : $T_a^\sigma(S) := \mathrm{Post}_a(S) \cap [\sigma]$
    $X := X \cup \{S\}$
endfor
$\mathrm{CPre}_1 := \emptyset$
while $X \neq \emptyset$
    choose $S \in X$, $X := X \setminus \{S\}$
    if $\exists a \in A_1 : a \in \mathrm{act}(S)$ and $\forall \sigma \in \mathrm{VIS}_1^V\ \exists U \in q : T_a^\sigma(S) \subseteq U$
        $\mathrm{CPre}_1 := \mathrm{CPre}_1 \cup \{S\}$
    else
        forall $v \in S$
            $S' := S \setminus \{v\}$, $X := X \cup \{S'\}$
            forall $(a,\sigma) \in A_1 \times \mathrm{VIS}_1^V$
                $M := \mathrm{Post}_a(\{v\}) \cap [\sigma]$
                $T_a^\sigma(S') := T_a^\sigma(S) \setminus \{u \in M \mid \forall w \in S' : (w,u) \notin E_a\}$
            endfor
        endfor
    endif
endwhile
output $\mathrm{CPre}_1$
```

Algorithm 5: Compute $\mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$

```
$X := \emptyset$, $\mathrm{Pre} := \mathrm{Pre}_{A_0}(\bigcup q) \cap V_0$
forall $\tau \in \mathrm{VIS}_1^V$
    $S := \mathrm{Pre} \cap [\tau]$
    forall $(a,\sigma) \in \overline{A} \times \mathrm{VIS}_1^V$ : $T_a^\sigma(S) := \mathrm{Post}_{[a]_{\sim_1}}(S) \cap [\sigma]$
    $X := X \cup \{S\}$
endfor
$\mathrm{CPre}_0 := \emptyset$
while $X \neq \emptyset$
    choose $S \in X$, $X := X \setminus \{S\}$
    if $\forall a \in \overline{A}\ \forall \sigma \in \mathrm{VIS}_1^V\ \exists U \in q : T_a^\sigma(S) \subseteq U$ : $\mathrm{CPre}_0 := \mathrm{CPre}_0 \cup \{S\}$
    else
        forall $v \in S$
            $S' := S \setminus \{v\}$, $X := X \cup \{S'\}$
            forall $(a,\sigma) \in \overline{A} \times \mathrm{VIS}_1^V$
                $M := \mathrm{Post}_{[a]_{\sim_1}}(\{v\}) \cap [\sigma]$
                $T_a^\sigma(S') := T_a^\sigma(S) \setminus \{u \in M \mid \forall w \in S'\ \forall b \in [a]_{\sim_1} : (w,u) \notin E_b\}$
            endfor
        endfor
    endif
endwhile
output $\mathrm{CPre}_0$
```

Proposition A.3.
(1) $\mathrm{CPre}_1 = \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$
(2) $\mathrm{CPre}_0 = \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_0^u$

Proof. (1) and (2) are proved completely analog so we just prove (1). First let $S \in \mathrm{CPre}_1$. Then $S \subseteq \mathrm{Pre}_{A_1}(\bigcup q) \cap V_1 \cap [\tau]$ for some $\tau \in \mathrm{VIS}_1^V$. Now the following condition is invariant under the while-loop over X. $T_a^\sigma(S) = \mathrm{Post}_a(S) \cap [\sigma]$ for all $(a,\sigma) \in A_1 \times \mathrm{VIS}_1^V$. This yields $S \in \mathrm{CPre}_1^{\subseteq}(q)$.

Now let $S \in \mathrm{CPre}_1^{\mathcal{A}}(q) \cap V_1^u$. Then obviously $S \subseteq \mathrm{Pre}_{A_1}(\bigcup q) \cap V_1 \cap [\tau] =: S'$ for some $\tau \in \mathrm{VIS}_1^V$. If $S = S'$ then obviously $S \in \mathrm{CPre}_1$. If $S \subsetneq S'$, then there are $x_1, \ldots, x_n \in S'$ such that $S = S' \setminus \{x_1, \ldots, x_n\}$. By induction over $i \leq n$ we show that after the i-th pass of the while-loop over X we have $S' \setminus \{x_1, \ldots, x_i\} \in X$. For $i = 1$ we just have to notice that by the maximality of S in $\mathrm{CPre}_1^{\subseteq}(q)$, the set S' is not in $\mathrm{CPre}_1^{\subseteq}(q)$. Now let $i > 1$. We know that $S' \setminus \{x_1, \ldots, x_{i-1}\}$ is in X and since S is maximal in $\mathrm{CPre}_1^{\subseteq}(q)$ we have $S' \setminus \{x_1, \ldots, x_{i-1}\} \notin \mathrm{CPre}_1^{\subseteq}(q)$. So the set $(S' \setminus \{x_1, \ldots, x_{i-1}\}) \setminus \{x_i\}$ is added to X in the i-th pass of the while-loop over X. Thus, in the n-th pass of the while-loop over X, we have $S \in X$ and from this we can infer that $S \in \mathrm{CPre}_1$. This also shows that each set $S \in \mathrm{CPre}_1$ is maximal in $\mathrm{CPre}_1^{\subseteq}(q)$ and so the proof is complete. □

Appendix B

A Note on Graph Complexity Measures

We have mentioned that the size of the equivalence classes of positions in a game with partial information certainly has some influence on the graph structure of the game graph of the corresponding game with full information. In particular, it influences certain measures of the graph complexity like tree-width, DAG-width and entanglement. Here we only want to see that if we do not bound the size of the equivalence classes, then the tree width of the game graph of the corresponding game with full information can be exponentially larger than the tree width of the original game graphs and vice versa. Since we are only dealing with the game graphs, we omit winning conditions in the descriptions of games here. First we have to introduce tree decompositions, tree-width and brambles.

Let $G = (V, E)$ be an undirected graph, that means, $E \subseteq V \times V$ is symmetric. A *tree decomposition* of G is a pair $(\mathcal{T}, \mathcal{X})$, where $\mathcal{T} = (T, S)$ is a tree and $\mathcal{X} : T \to 2^V$ is a function assigning to each node of T a subset of V, such that the following conditions hold.

(T1) $V = \bigcup\{\mathcal{X}(t) \,|\, t \in T\}$.

(T2) For each edge $\{u, v\} \in E$ there is some $t \in T$ such that $\{u, v\} \subseteq \mathcal{X}(t)$.

(T3) For all $v \in V$, the set $\{t \in T \,|\, v \in \mathcal{X}(t)\}$ induces a connected subtree of $\mathcal{T}$.

The *width* of $(\mathcal{T}, \mathcal{X})$ is defined by $\max_{t\in T} |\mathcal{X}(t)| - 1$. The *tree width* of G, denoted by $\mathrm{tw}(G)$ is the minimum width of all possible tree decompositions of G.

Two subsets $U_1, U_2 \subseteq V$ are called *touching*, if $U_1 \cap U_2 \neq \emptyset$ or if there are $u_1 \in U_1$ and $u_2 \in U_2$ such that $\{u_1, u_2\} \in E$. A set $U \subseteq V$ *covers* a set $\mathcal{U} \subseteq 2^V$, if for each $U' \in \mathcal{U}$ we have $U' \cap U \neq \emptyset$. A *bramble* in G is a set of connected, mutually touching subsets of V. The minimum size of all sets covering a bramble is called the *order* of the bramble.

Now the following statements are equivalent.

- G has a bramble of order at least k.
- G has tree-width at least $k - 1$.

The tree width of a *directed graph* is the tree width of the underlying undirected graph, that means, the graph with the same set of nodes and where the edge relation is the symmetric closure of the original edge relation. Finally, the tree width of a game is the tree width of its underlying game graph.

Proposition B.1. *There is a sequence* $(\mathcal{G}^n)_{n<\omega}$ *of games with partial information and some designated initial position* v_0*, such that for each* $n < \omega$*,* $\mathcal{G}^n$ *has tree-width* 1 *and the corresponding game* $\overline{G}^n_{v_0}$ *with full information has tree width at least* n*.*

Proof. For $n < \omega$ let $\mathcal{G}^n = (G^n, (\mathrm{vis}^V_{i,n}), (\mathrm{vis}^A_{i,n}))$ with $G^n = (V^n, V^n, (f^n_a)_{a\in A^n})$ be defined as follows.

- $V^n = \{1\} \cup \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$
- $A^n = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$
- $f^n_{a_i}(1) = u_i$ and $f^n_{a_i}(u_i) = 1$ for $i \in \{1, \dots, n\}$
- $f^n_{b_i}(u_i) = v_i$ and $f^n_{b_i}(v_i) = u_i$ for $i \in \{1, \dots, n\}$
- $\mathrm{vis}^V_{1,n}(v_i) = \mathrm{vis}^V_{1,n}(1) = 1$ and $\mathrm{vis}^V_{1,n}(u_i) = u_i$ for $i \in \{1, \dots, n\}$.
- $\mathrm{vis}^A_{1,n}(a_i) = \mathrm{vis}^A_{1,n}(b_i) = 1$ for all $i \in \{1, \dots, n\}$.

Furthermore, $v_0 = 1$ is the initial position. The underlying undirected graph of the game graph of G^n is a tree, so G^n has tree width 1.

Moreover, for $n < \omega$, the corresponding game $\overline{G}^n_1 = (\overline{V}^n, \overline{V}^n, (\overline{E}^n_a)_{a\in\overline{A}^n})$ with full information has the following components.

- $\overline{V}^n = \{\{1\}\} \cup \{\{u_1\}, \dots, \{u_n\}\} \cup \{\{1, v_1\}, \dots, \{1, v_n\}\}$
- $\overline{A}^n = \{a\}$
- $\overline{E}_a$ is the union of the following sets.
  - $\{(\{1\}, \{u_i\}) \mid i = 1, \dots, n\}$
  - $\{(\{u_i\}, \{1, v_i\}) \mid i = 1, \dots, n\}$
  - $\{(\{1, v_i\}, \{u_j\}) \mid i, j = 1, \dots, n\}$

So it is easy to see that the set $\mathcal{U} = \{\{\{u_1\}\}, \{\{1, v_1\}\}, \{\{u_2\}, \{1, v_2\}\}, \dots, \{\{u_n\}, \{1, v_n\}\}\}$ is a bramble of order $n+1$ in the game graph of $\overline{G}^n_1$ and so, the tree width of $\overline{G}^n_1$ is at least n. ☐

Remark. It is easy to see that $\overline{G}^n_1$ has tree width exactly n.

Proposition B.2. *There is a sequence* $(\mathcal{G}^n)_{n<\omega}$ *of games with partial information and some designated initial position* v_0*, such that for each* $n < \omega$*,* $\mathcal{G}^n$ *has tree width* $n-1$ *and the corresponding game* $\overline{G}^n_{v_0}$ *with full information has tree width* 1*.*

Proof. For $n < \omega$ let $\mathcal{G}^n = (G^n, (\mathrm{vis}^V_{i,n}), (\mathrm{vis}^A_{i,n}))$, $G^n = (V^n, V^n, (f^n_i)_{i\in A^n})$ be defined as follows.

- $V^n = \{1, \ldots, n\}$.
- $A^n = \{1, \ldots, n\}$.
- $f_i^n(j) = i$ for $i, j \in \{1, \ldots, n\}$.
- $\mathrm{vis}^V_{1,n}(i) = 1$ and $\mathrm{vis}^A_{1,n}(i) = 1$ for $i \in \{1, \ldots, n\}$.

Furthermore, $v_0 = 1$ is the initial position. The underlying undirected graph of the game graph of G^n is the complete graph with n nodes, so G^n has tree width $n - 1$.

Moreover, for $n < \omega$, the corresponding game $\overline{G}^n_{v_0}$ with full information can be represented as follows.

$$\{1\} \xrightarrow{1} \{1, \ldots, n\} \overset{1}{\circlearrowleft}$$

So obviously, $\overline{G}^n_{v_0}$ has tree width 1. $\square$