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Ehrenfeucht-Fraïssé Games for Semiring Semantics

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Abstract

Semiring semantics for first-order logic generalizes classical semantics by extending the domain of truth values to an arbitrary commutative semiring $\mathcal{K}$. By this means, the evaluation of a formula on a $\mathcal{K}$-interpretation provides additional information targeting questions such as why or to what extent the formula is satisfied.

Since Boolean semantics occurs as a particular semiring choice, the question arises as to which classical results can be extended to semiring semantics. This thesis is concerned with the transferability of the Ehrenfeucht-Fraïssé game, which is of central importance in classical model theory, as it provides a convenient method to decide whether any two structures are distinguishable by a first-order formula up to a certain quantifier rank.

We show that the direct translation of the game rules to $\mathcal{K}$-interpretations does not yield a characterization of m-equivalence under semiring semantics for any semiring unless it is isomorphic to the Boolean semiring. In particular, we identify full idempotence as an algebraic characterization of the class of semirings for which the m-turn Ehrenfeucht-Fraïssé game is a sound proof method for m-equivalence. Thus, we present two alternative notions of equivalence which are characterized by the m-turn Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations instead. As opposed to the classical Ehrenfeucht-Fraïssé game, the m-turn counting game is sound with respect to any semiring, and the cardinalities of the selectable sets can be bounded based on the semiring properties. While not complete in general, we show that the m-turn counting game characterizes m-equivalence with respect to the natural semiring $\mathbb{N}$. Finally, we discuss further modifications of the game rules and derive a game which characterizes m-equivalence under semiring semantics for distributive lattices.

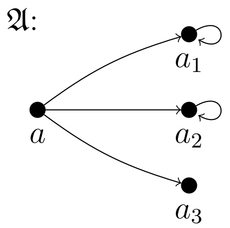
Contents

Chapter 1

Introduction

In classical first-order logic, any sentence φ is either satisfied by a structure $\mathfrak{A}$ or not satisfied by $\mathfrak{A}$, so there are only two different truth values. However, not only the fact whether $\mathfrak{A}$ is a model of φ might be of interest but additional practical information concerning the evaluation of φ on $\mathfrak{A}$. For instance, it can be useful to track the number of evaluation strategies for establishing $\mathfrak{A} \models \varphi$, the confidence one has that $\mathfrak{A} \models \varphi$ holds, or the cost or clearance level required for the evaluation of φ on $\mathfrak{A}$. This raises the question whether the classical semantics of first-order logic can be generalized such that additional information can be encoded via multiple truth values beyond true and false.

As an example, consider the evaluation of $\varphi(x) = \exists y(Exy \wedge Eyy)$ on the structure $\mathfrak{A}$, where the free variable x is instantiated with a and E is a binary relation defined according to the edges.

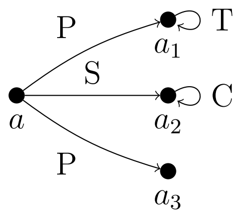


$$
\begin{array}{rlcccll}
 & & Eay & & Eyy & & \\
[\![\varphi(a)]\!]^{\mathfrak{A}} = & (& 0 & \wedge & 0 &) & y \mapsto a \\
\vee & (& 1 & \wedge & 1 &) & y \mapsto a_1 \\
\vee & (& 1 & \wedge & 1 &) & y \mapsto a_2 \\
\vee & (& 1 & \wedge & 0 &) & y \mapsto a_3 \qquad = 1
\end{array}
$$

The evaluation of φ on $\mathfrak{A}$ only depends on the truth values of the atomic properties determined by $\mathfrak{A}$, which are propagated to φ using the binary functions $\vee$ and $\wedge$ on $\{0, 1\}$. In this manner, the evaluation of a classical first-order formula on a structure can be considered an algebraic computation, which suggests to extend the domain of truth values by replacing the two-element Boolean algebra by more complex algebraic structures.

This approach was implemented in [GT17a] as *semiring semantics for first-order logic*. In this generalization of classical semantics, first-order formulae are evaluated on $\mathcal{K}$-interpretations instead of classical structures, which map instantiated literals such as Eaa and $\neg Eaa$ into a commutative semiring $\mathcal{K} = (K, +, \cdot, 0, 1)$ rather than assigning a Boolean truth value. Whereas the identity element with regard to addition, the 0, reflects falsity, any other semiring element is intended to represent some nuance of truth. This way, the semantics of a first-order formula is obtained by inductively applying the semiring operations to the basic valuations defined by the $\mathcal{K}$-interpretation. Thereby, we keep track of whether information is used alternatively, as in the case of disjunctions and existential quantifications, which is reflected by addition, or jointly in a conjunction or universal quantification, corresponding to multiplication. By contrast, the semantics of negation cannot be linked to an operation inherent in the semiring, which is why semiring semantics, as defined in [GT17a], refers to formulae in negation normal form only.

The access control semiring $\mathbb{A} = (\{P, C, S, T, 0\}, \min, \max, 0, P)$ constitutes a possible application semiring which can be used to track certain clearance levels required to access the information modeled by an interpretation. The linearly ordered semiring elements $P < C < S < T < 0$ can be interpreted as clearance levels representing **p**ublic, **c**onfidential, **s**ecret and **t**op secret, whereas 0 refers to false or inaccessible. Moving from the initial classical structure $\mathfrak{A}$ to an $\mathbb{A}$-interpretation enables an annotation of the true atomic facts with some clearance level, for instance as depicted below. We omit valuations of negated literals in the drawing, as they do not influence the semantics of the formula $\varphi(a) = \exists y(Eay \wedge Eyy)$.



$$
\begin{array}{lll}
 & \quad Eay \;\; Eyy & \\
\min(& \max(\,0,\;\; 0\,), & y \mapsto a \\
 & \max(\,P,\;\; T\,), & y \mapsto a_1 \\
 & \max(\,S,\;\; C\,), & y \mapsto a_2 \\
 & \max(\,P,\;\; 0\,)) & y \mapsto a_3 \qquad = S
\end{array}
$$

Using the addition and multiplication in $\mathbb{A}$, which is defined by minimum and maximum, we can conclude that the clearance level S is required to prove that $\varphi(a)$ is satisfied by $\mathfrak{A}$.

Semiring semantics does not only provide the opportunity to encode practical information in the interpretations such as clearance levels, confidence scores or costs and to take it into account when evaluating first-order formulae. Beyond that, the approach can also be used to gain a better understanding of why a formula is satisfied by a classical structure. By using appropriate semirings of polynomials or formal power series and interpreting the atomic properties which are satisfied in some structure $\mathfrak{A}$ by unique variables, the valuation of a sentence φ reveals which combinations of atomic properties imply that $\mathfrak{A} \models \varphi$. In this manner, it is possible to obtain more detailed information on how the outcome of the model checking computation is affected by the atomic facts satisfied by $\mathfrak{A}$. Trying to understand the origin of the output of a process such as the model checking problem is generally referred to as *provenance analysis*. The idea to use semirings for this purpose originated in database theory with the aim of tracking why certain tuples of a relational database satisfy a given query (see e.g. [GKT07, GT17b]). Recently, semiring provenance was not only applied to first-order logic but to various other logics including guarded logics, description logics as well as fixed-point logics [DG21, BOPP20, DG19, DGNT21].

Considering semiring semantics for first-order logic as a generalization of classical semantics raises the question which classical results, for instance from model theory, are preserved when extending the domain of truth values. As an example, every finite classical structure can be defined up to isomorphism in FO, while elementary equivalence of finite $\mathcal{K}$-interpretations does not imply isomorphism in general. This observation, established in [GM21], indicates that the model-theoretic properties of semiring semantics differ significantly from those of classical semantics and strongly depend on the algebraic properties of the underlying semiring.

In this thesis, we examine the transferability of the Ehrenfeucht-Fraïssé game, as a proof method for m-equivalence, to $\mathcal{K}$-interpretations and semiring semantics. According to the Ehrenfeucht-Fraïssé theorem, the problem of deciding whether two classical structures can be distinguished by a sentence of quantifier rank at most m can be reduced to a game which is played by two players who pick elements from the universes of the structures in question. By this means, it suffices to compare the local atomic properties of the structures at the end of each play. The game relies on the observation that $\mathfrak{A} \models \exists x \psi(x)$ if, and only if, there is a witness a in the universe of $\mathfrak{A}$ such that $\mathfrak{A} \models \psi(a)$. This makes it possible to localize the potential differences between two given classical structures. By contrast, in semiring semantics the valuation of $\exists x \psi(x)$ is obtained by summing up the valuations of the instantiated subformulae $\psi(a)$ and can thus not be attributed to a single element. While distinct families of summands may lead to the same sum, different sums might be obtained even if the sets of summands coincide, as certain summands may occur several times. This observation provides first insights that the application of the Ehrenfeucht-Fraïssé game to $\mathcal{K}$-interpretations poses problems, as the quantifiers in semiring semantics deviate fundamentally from Boolean quantifiers. But before examining the transferability of the Ehrenfeucht-Fraïssé game more closely, we first introduce $\mathcal{K}$-interpretations as well as semiring semantics formally and summarize the required algebraic foundations.

Chapter 2

Preliminaries

Prior to extending the semantics of classical first-order logic to multiple different levels of truth providing insights into how or why a formula is satisfied, we will first identify commutative semirings as suitable algebraic structures for generalizing Boolean truth values. More precisely, we are aiming for algebraic structures with two operations modeling the alternative or joint use of information, respectively. Moreover, a distinguished element is supposed to reflect false assertions, whereas multiple other elements are intended to model different nuances of truth.

2.1 Commutative Semirings

In order to define commutative semirings formally, we make use of the notion of *commutative monoids* which are defined as sets equipped with a commutative, associative binary operation and an identity element.

Definition 2.1. A *commutative semiring* is an algebraic structure $\mathcal{K} = (K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids with $0 \neq 1$,

(1) multiplication by 0 annihilates K, i.e., $k \cdot 0 = 0$ for all $k \in K$ and

(2) multiplication distributes over addition, i.e., $(k_1 + k_2) \cdot k_3 = k_1k_3 + k_2k_3$ for all $k_1, k_2, k_3 \in K$.

As all semirings considered within this thesis are commutative, we will implicitly assume commutativity in the following and refer to commutative semirings just as semirings for convenience.

The semiring axioms constitute reasonable assumptions for the purpose of extending the domain of truth values, which arises from considering the algebraic contraints as logical equivalences. In this manner, commutativity of multiplication, for example, represents the condition that formulae $\varphi \wedge \psi$ and $\psi \wedge \varphi$ are intended to be equivalent, while the requirement that 0 is annihilating can be interpreted as the assumption that the conjunction with some false fact always needs to result in a false assertion as well.

Furthermore, there are several applications which can be modeled by semirings and provide useful practical information about the evaluation of a formula.

- Using the *Boolean semiring* $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ enables valuations by classical truth values and results in classical semantics. Thus, semiring semantics indeed generalizes Boolean semantics, which occurs as a particular semiring.

- The *access control semiring* $\mathbb{A} = (\{P, C, S, T, 0\}, \min, \max, 0, P)$ allows assignments of the true atomic facts with one of the clearance levels **p**ublic, **c**onfidential, **s**ecret and **t**op secret, where $P < C < S < T < 0$, and argue about access levels required for establishing that a sentence is satisfied.

- Elements of the *Viterbi semiring* $\mathbb{V} = ([0,1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ can be used to label the atomic facts by a degree of trust and reason about confidence.

- The *tropical semiring* $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ provides the opportunity to annotate basic facts with a cost which has to be paid for accessing them and realize a cost analysis. It is isomorphic to $\mathbb{V}$ via the bijective mappings $x \mapsto e^{-x}$ and $y \mapsto -\ln y$.

- An alternative semiring to annotate the atomic facts with confidence scores is given by the *Łukasiewicz semiring* $\mathbb{L} = ([0,1]_{\mathbb{R}}, \max, \odot, 0, 1)$, where $k \odot \ell = \max(k + \ell - 1, 0)$. Its isomorphic variant $\mathbb{D} = ([0,1]_{\mathbb{R}}, \min, \oplus, 1, 0)$ with $k \oplus \ell = \min(k + \ell, 1)$ can be used to reason about doubt.

- The *natural semiring* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ can, for instance, be used to track the number of evaluation strategies proving that a certain sentence is satisfied.

There are multiple other equivalences such as $\varphi \vee \varphi \equiv \varphi$ or $\varphi \vee (\varphi \wedge \psi) \equiv \varphi$ which clearly hold in classical semantics but are not covered by the semiring axioms. The corresponding algebraic properties ensuring that these equivalences are true in semiring semantics as well lead to a further classification of semirings. Thus, the more of the following properties are satisfied by a certain semiring, the more similarly to classical semantics this semiring behaves.

Definition 2.2. A semiring $\mathcal{K}$ is called

- *idempotent* if $k + k = k$ for all $k \in K$,
- *multiplicatively idempotent* if $k \cdot k = k$ for all $k \in K$,
- *fully idempotent* if $\mathcal{K}$ is both idempotent and multiplicatively idempotent and
- *absorptive* if $k + k\ell = k$ for all $k, \ell \in K$.

Since idempotence appears as a special case of absorption where $\ell = 1$, every absorptive semiring is also idempotent.

While requiring the semiring axioms can be justified by linking them to the logical equivalences they represent, the question arises as to why we do not consider more specific algebraic structures such as rings or fields. It turns out that the existence of inverse elements is incompatible with the intention that all elements except for 0 model some level of truth. If some element $k \neq 0$ was invertible with regard to addition by $-k$ and two assertions were valuated with k and $-k$, respectively, we would expect both facts to be true, whereas the disjunction would be valuated $k + (-k) = 0$ and hence considered false. As we want any disjuction of true facts to result in a true fact as well, we are particularly interested in semirings, where not a single non-zero element can be inverted. The analogous can be observed for the existence of inverse elements with regard to multiplication and conjunctions. As we will later see, the absence of inverse elements in *positive* semirings ensures the desired duality of negation.

Definition 2.3. A semiring $\mathcal{K}$ is called *positive*, if for all elements $k, \ell \in K$

- $k + \ell = 0$ implies $k = 0$ and $\ell = 0$ and
- $k \cdot \ell = 0$ implies $k = 0$ or $\ell = 0$.

In different contexts, it is required to compare certain semiring valuations, which is why an order on the semiring elements needs to be associated with every semiring. Order theory plays a crucial role in defining semiring semantics for fixed-point logics [GT20, DGNT21], but finds application in semiring semantics for first-order logic as well. While the binary semiring operations only induce finite summations and multiplications of semiring elements, infinitary extensions are required to evaluate the semantics of quantifiers over infinite universes. This is mostly realized by considering the supremum or infimum of the finite subsums or subproducts which demands to define an order on the semiring elements. Intuitively, the order is supposed to reflect that a certain elements is “at most as true” as some other element, which is why we make use of the order induced by addition, as we expect φ to be at most as true as $\varphi \vee \psi$ and 0 to be the minimum truth level.

Definition 2.4. A semiring $\mathcal{K}$ is *naturally ordered*, if the order $\leq$ on $\mathcal{K}$, defined by

$$k \leq \ell \text{ if, and only if, there is some } k' \text{ with } k + k' = \ell,$$

is a partial order, i.e., reflexive, transitive and antisymmetric. In this case, we refer to $\leq$ as the *natural order* on $\mathcal{K}$.

While reflexivity and transitivity is fulfilled in every semiring, this is not true for antisymmetry. For instance, if an element k was invertible with some distinct $-k$, we obtain that $k \leq -k$ and $-k \leq k$, although $k \neq -k$ by assumption, which again illustrates why we do not use rings instead of semirings. The advantage of defining the natural order based on addition instead of multiplication is given by the monotonicity of both operations, which follows from distributivity.

Lemma 2.5. On any naturally ordered semiring $\mathcal{K}$, addition and multiplication are monotone, i.e., for all $k_1, k_2, \ell_1, \ell_2 \in K$ with $k_1 \leq \ell_1$ and $k_2 \leq \ell_2$ it holds that $k_1 + k_2 \leq \ell_1 + \ell_2$ and $k_1 \cdot k_2 \leq \ell_1 \cdot \ell_2$.

Proof. Since $k_1 \leq \ell_1$ and $k_2 \leq \ell_2$ by assumption, there must be elements $k_1', k_2' \in K$ with $k_1 + k_1' = \ell_1$ and $k_2 + k_2' = \ell_2$. It follows that

$$\begin{aligned}
k_1 + k_2 &\leq (k_1 + k_2) + (k_1' + k_2') \\
&= (k_1 + k_1') + (k_2 + k_2') = \ell_1 + \ell_2 \text{ and} \\
k_1 \cdot k_2 &\leq (k_1 k_2) + (k_1 k_2' + k_1' k_2 + k_1' k_2') \\
&= (k_1 k_2 + k_1 k_2') + (k_1' k_2 + k_1' k_2') \\
&= k_1(k_2 + k_2') + k_1'(k_2 + k_2') \\
&= (k_1 + k_1') \cdot (k_2 + k_2') = \ell_1 \cdot \ell_2.
\end{aligned}$$

□

However, distributivity, as the only semiring axiom linking the natural order with multiplication, is not sufficient for ensuring that the valuation of $\varphi \wedge \psi$ is smaller than or equal to the valuation of φ. But in several applications it is reasonable to assume that any conjunction is at most as true as the single subformulae it consists of. This compatibility of the natural order with multiplication is ensured by requiring absorption, which is why absorptive semirings are of particular interest for semiring semantics.

Lemma 2.6. Any naturally ordered semiring $\mathcal{K}$ is absorptive if, and only if, multiplication is decreasing in $\mathcal{K}$, i.e., $k \cdot \ell \leq k$ for all $k, \ell \in K$.

Proof. ($\Rightarrow$) Suppose that $\mathcal{K}$ is absorptive. For any $k \in K$ it holds that $1+1\cdot k = 1$, thus $k \cdot \ell \leq \ell + k \cdot \ell = (1+k) \cdot \ell = 1 \cdot \ell = \ell$ for all $\ell \in K$.

($\Leftarrow$) If multiplication is decreasing in $\mathcal{K}$, we have that $k + k \cdot \ell = k \cdot (1+\ell) \leq k$ for all $k, \ell \in K$. Further $k \leq k + k \cdot \ell$ is true by definition, hence the antisymmetry of $\leq$ yields $k = k + k \cdot \ell$ and $\mathcal{K}$ is absorptive. $\square$

2.2 Semiring Semantics for First-Order Logic

Having established the algebraic basics, we will now give a formal definition of semiring semantics and the corresponding interpretations the first-order formulae are evaluated on according to [GT17a]. Given a finite relational vocabulary τ and a non-empty universe A, we define the set of *τ-literals over A* as

$$\mathrm{Lit}_A(\tau) := \{R\bar{a}, \neg R\bar{a} \colon R \in \tau \text{ and } \bar{a} \in A^{\mathrm{arity}(R)}\}.$$

As (in)equalities are treated differently compared to the literals referring to a relation in semiring semantics, we assume that (in)equalities are not contained in the set of τ-literals over A. In case (in)equalities are supposed to be considered as well, we refer to the set of *literals (over A)* in $\mathrm{FO}(\tau)$ as usual. For simplicity, we assume that equivalences such as $\neg\neg R\bar{a} \equiv R\bar{a}$ or $\neg\neg x = x \equiv x = x$ are identified and therefore suppose that the negation of a literal (over A) is again a literal (over A).

As for the syntax, we consider the same formulae as in classical first-order logic and thus transfer the basic notions referring to the syntax such as *quantifier rank*, which we denote as $\mathrm{qr}(\varphi)$, or *negation normal form*, denoted by $\mathrm{nnf}(\varphi)$, for any formula $\varphi \in \mathrm{FO}(\tau)$.

Unlike for Boolean semantics, the first-order formulae will not be evaluated on classical structures $\mathfrak{A} = (A, \tau)$. A classical structure interprets the relation symbols by determining which tuples of elements are contained in a certain relation and which are not, so by defining some function $\mathrm{Lit}_A(\tau) \to \mathbb{B}$. We generalize this idea by considering functions $\mathrm{Lit}_A(\tau) \to \mathcal{K}$ mapping into a semiring $\mathcal{K}$ and therefore demand interpretations to valuate every literal with a semiring element.

Definition 2.7. Let $\mathcal{K}$ be a semiring, τ a finite relational vocabulary and A a non-empty universe. A *$\mathcal{K}$-interpretation* is a mapping $\pi \colon \mathrm{Lit}_A(\tau) \to \mathcal{K}$.

As several examples of $\mathcal{K}$-interpretations we consider later on are based on vocabularies that only consist of unary relation symbols, we will denote them in the following manner.

	A	R_1	R_2	$\dots$	$\neg R_1$	$\neg R_2$	$\dots$
$\pi :$	a	$\pi(R_1 a)$	$\pi(R_2 a)$	$\dots$	$\pi(\neg R_1 a)$	$\pi(\neg R_2 a)$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In order to define the semantics, we inductively extend this mapping to arbitrary formulae in $\mathrm{FO}(\tau)$, making use of the idea that the valuations of complex formulae arise from the valuations of the subformulae by applying the algebraic operations. Hence, the valuations of the subformulae a certain formula consists of are supposed to completely determine the valuation of this formula. This idea cannot directly be transferred to negation, constituting a non-compositional operation in the semiring setting: As soon as we consider a second level of truth next to 1, just based on the valuation of some formula φ, it is not clear how $\neg\varphi$ should be valuated if φ is assigned 0. In order to cope with this issue, we assume that all first-order formulae are given in negation normal form, i.e., negations only occur in front of atomic formulae. For all remaining formulae φ, we simply assign the semantics of $\mathrm{nnf}(\varphi)$. The fact that every first-order formula can be transferred into a formula in negation normal form which is equivalent in classical semantics by making use of the duality of the quantifiers and applying the De Morgan's laws justifies this assumption.

Definition 2.8. A $\mathcal{K}$-interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{K}$ can inductively be extended to formulae $\varphi(x_1, \dots, x_n) \in \mathrm{FO}(\tau)$ in negation normal form, given a variable assignment $\beta\colon X \to A$ with $\{x_1, \dots, x_n\} \subseteq X$ as follows:

Case 1 ((in)equalities). For $\varphi(\bar{x}) = x_{i_1} \circ x_{i_2}$ with $\circ \in \{=, \neq\}$ and $1 \leq i_1, i_2 \leq n$ we set

$$\pi[\![x_{i_1} = x_{i_2}]\!]^\beta = \begin{cases} 1, & \beta(x_{i_1}) = \beta(x_{i_2}) \\ 0, & \beta(x_{i_1}) \neq \beta(x_{i_2}) \end{cases} \quad \text{and}$$

$$\pi[\![x_{i_1} \neq x_{i_2}]\!]^\beta = \begin{cases} 0, & \beta(x_{i_1}) = \beta(x_{i_2}) \\ 1, & \beta(x_{i_1}) \neq \beta(x_{i_2}) \end{cases} .$$

Case 2 (τ-literals). If $\varphi(\bar{x}) \in \{Rx_{i_1} \dots x_{i_r}, \neg Rx_{i_1} \dots x_{i_r}\}$, where $R \in \tau$, $\mathrm{arity}(R) = r$ and $1 \leq x_{i_1}, \dots, x_{i_r} \leq n$, then

$$\pi[\![Rx_{i_1} \dots x_{i_r}]\!]^\beta = \pi(R\beta(x_{i_1}) \dots \beta(x_{i_r})) \text{ and}$$
$$\pi[\![\neg Rx_{i_1} \dots x_{i_r}]\!]^\beta = \pi(\neg R\beta(x_{i_1}) \dots \beta(x_{i_r})).$$

Case 3 (disjunction, conjunction). For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$, where $\circ \in \{\vee, \wedge\}$, set

$$\pi[\![\psi(\bar{x}) \vee \vartheta(\bar{x})]\!]^\beta = \pi[\![\psi(\bar{x})]\!]^\beta + \pi[\![\vartheta(\bar{x})]\!]^\beta \text{ and}$$
$$\pi[\![\psi(\bar{x}) \wedge \vartheta(\bar{x})]\!]^\beta = \pi[\![\psi(\bar{x})]\!]^\beta \cdot \pi[\![\vartheta(\bar{x})]\!]^\beta.$$

Case 4 (quantifiers). For $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ we define

$$\pi[\![\exists x\psi(\bar{x}, x)]\!]^{\beta} = \sum_{a\in A} \pi[\![\psi(\bar{x}, x)]\!]^{\beta[x/a]} \text{ and}$$
$$\pi[\![\forall x\psi(\bar{x}, x)]\!]^{\beta} = \prod_{a\in A} \pi[\![\psi(\bar{x}, x)]\!]^{\beta[x/a]},$$

where the function $\beta[x/a]$ maps all variables to the same element as β except for x which is mapped to a. To simplify notation we will use $\pi[\![\varphi(a_1, \dots, a_n)]\!]$ in the following to denote $\pi[\![\varphi(x_1, \dots, x_n)]\!]^{\beta}$, where $\beta(x_i) = a_i$ for $1 \leq i \leq n$.

In the approach defined in [GT17a], equalities are intended to be untracked in provenance analysis, which is why equalities are not valuated by $\mathcal{K}$-interpretations. Instead, their semantics is defined analogous to the Boolean case by assigning the neutral elements, independent of the interpretation the equality is evaluated on.

The idea behind the semantics of the quantifiers is that they can be regarded as a disjunction or conjunction, respectively, over all elements a of the universe A, for which the corresponding subformula is evaluated. In case of finite A, the semantics of quantifiers is well-defined, as the binary semiring operations can be applied repeatedly. Thereby, associativity and commutativity of $+$ and $\cdot$ justifies not respecting a certain order of the elements in the universe and using the notation $\sum_{a\in A}$ and $\prod_{a\in A}$. However, we want to permit infinite $\mathcal{K}$-interpretations, that is, $\mathcal{K}$-interpretations with infinite universe, as well. In this case summing or multiplying over all elements poses a problem, as infinite sums and products are not inherent in the semiring, so, in general, there are multiple ways to define the infinitary operations. Analogous to the semiring axioms which determine the fundamental behavior of finite sums and products, we formulate certain requirements on the infinite sums and products as well, as proposed in [Mrk20].

Definition 2.9. A semiring $\mathcal{K}$ *admits infinitary summation*, if there is a summation operator Σ such that for all index sets I, J and elements $(k_i)_{i\in I} \in K^I, \ell \in K$

(1) Σ respects finite sums: $\sum_{i\in I} k_i = \sum_{i\in I}^{\text{fin}} k_i$, if I is finite

(2) Σ is invariant under bijections: $\sum_{i\in I} k_i = \sum_{i\in I} k_{\sigma(i)}$ for all permutations σ of I

(3) Σ is invariant under partitions: $\sum_{i\in I} k_i = \sum_{S\in\mathcal{P}} \sum_{i\in S} k_i$ for all partitions $\mathcal{P}$ of I

(4) Multiplication distributes over Σ: $\ell \cdot \sum_{i\in I} k_i = \sum_{i\in I} \ell \cdot k_i$

Analogously, $\mathcal{K}$ is said to *admit infinitary multiplication*, if there is a multiplication operator Π such that for all index sets I, J and elements $(k_i)_{i\in I} \in K^I, \ell \in K$

(1) Π respects finite sums: $\prod_{i\in I} k_i = \prod_{i\in I}^{\text{fin}} k_i$, if I is finite

(2) Π is invariant under bijections: $\prod_{i\in I} k_i = \prod_{i\in I} k_{\sigma(i)}$ for all permutations σ of I

(3) Π is invariant under partitions: $\prod_{i\in I} k_i = \prod_{S\in\mathcal{P}} \sum_{i\in S} k_i$ for all partitions $\mathcal{P}$ of I

We overall say that $\mathcal{K}$ *admits infinitary operations*, if it admits both infinitary summation and infinitary multiplication.

While (1) makes sure that the infinitary operations are indeed a generalization of the addition and multiplication which are defined by the semiring, invariance under bijections and partitions can be construed as infinitary versions of commutativity and associativity. In the same manner as we expect multiplication to distribute over summation as a binary function and obtain the equivalence $\varphi \wedge \exists x\psi(x) \equiv \exists x(\varphi \wedge \psi(x))$ for finite universes, property (4) ensures that this equivalence holds in case of an infinite universe as well.

The conditions on the infinitary operations defined above cannot be realized in every semiring. As an example, suppose that the natural semiring $\mathbb{N}$ admitted infinitary summation. Then, $\sum_{i\in\mathbb{N}} 1$ would have to be mapped to some natural number n, but invariance under partitions requires

$$n = \sum_{i\in\mathbb{N}} 1 = 1 + \sum_{i\in\mathbb{N}\setminus\{0\}} 1 = 1 + \sum_{i\in\mathbb{N}} 1 = 1 + n,$$

which is not satisfied by any natural number n. Therefore, we only consider infinite $\mathcal{K}$-interpretations if $\mathcal{K}$ admits infinitary operations. Whenever this is the case we will assume some arbitrary, but fixed infinitary operations without denoting them explicitly. If we refer to a specific semiring, we explicitly define the considered infinitary operations, which mostly emerge from the finite subsums or subproducts by considering the supremum or infimum with regard to the natural order.

The definition of the basic properties of the binary semiring operations such as idempotence, positivity or monotonicity can easily be extended to the infinitary operations. As an example, positivity can be transferred to infinitary operations Σ and Π by demanding for all $(k_i)_{i\in I} \in K^I$ and index sets I that

- $\sum_{i\in I} k_i = 0$ implies $k_i = 0$ for all $i \in I$ and

- $\prod_{i \in I} k_i = 0$ implies $k_i = 0$ for some $i \in I$.

In general, these properties do not necessarily have to hold for the infinitary operations, if they are true in the binary case. Pertaining to positivity, this can, for instance, be observed for the Viterbi semiring. The most straightforward way to define the infinitary operations for $\mathbb{V}$-interpretations is to consider the supremum of the finite subsums and the infimum of the finite subproducts with respect to the natural order, that is,

$$\sum_{i \in I} k_i := \sup_{\substack{I' \subseteq I \\ \text{finite}}} \sum_{i \in I'}^{\text{fin}} k_i \quad \text{and} \quad \prod_{i \in I} k_i := \inf_{\substack{I' \subseteq I \\ \text{finite}}} \prod_{i \in I'}^{\text{fin}} k_i$$

for all families $(k_i)_{i \in I} \in \mathbb{V}^I$ over index set I. As an example, for the family $(k_i)_{i \in \mathbb{N}}$ of elements in $\mathbb{V}$ with $k_i = 0.5$ for all $i \in \mathbb{N}$, we observe that

$$\prod_{i \in \mathbb{N}} 0.5 = \inf_{\substack{I' \subseteq I \\ \text{finite}}} \prod_{i \in I'}^{\text{fin}} 0.5 = 0,$$

although $k_i > 0$ for all $i \in \mathbb{N}$. Hence, the infinitary operations do not maintain the positivity of the binary operations in $\mathbb{V}$, which illustrates that it is not possible in general to extrapolate from the finite to the infinitary operations as regards properties such as positivity.

To be precise, different summation and multiplication operations are used when evaluating the semantics of a formula on a certain $\mathcal{K}$-interpretation π, depending on whether π is finite or not. However, we will assume for convenience that whenever we fix a $\mathcal{K}$-interpretation π over a semiring with a certain property, then this property has to be satisfied by the finite operations only, if π is finite and by their infinitary extensions in the case of infinite π.

In order to illustrate that the classical semantics of first-order logic indeed appears as a special case of semiring semantics, a $\mathbb{B}$-interpretation $\pi_{\mathfrak{A}}$ can be associated with each classical structure $\mathfrak{A}$ by taking the universe A from $\mathfrak{A}$ and setting $\pi_{\mathfrak{A}}(L) = 1$ if, and only if, $\mathfrak{A} \models L$ for all $L \in \mathrm{Lit}_A(\tau)$. It can be shown that $\pi_{\mathfrak{A}}$ exhibits the same behavior as $\mathfrak{A}$ in the following sense.

Proposition 2.10. For any structure $\mathfrak{A} = (A, \tau)$, $\bar{a} = (a_1, \dots, a_n) \in A^n$ and any formula $\varphi(\bar{x})$ it holds that

$$\mathfrak{A} \models \varphi(\bar{a}) \text{ if, and only if, } \pi_{\mathfrak{A}}[\![\varphi(\bar{a})]\!] = 1.$$

For a converse sanity check, we associate classical structures with $\mathcal{K}$-interpretations by omitting the precise truth values and only distinguishing between zero and non-zero valuations. However, this presumes that every literal or its negation is interpreted with 0 by the $\mathcal{K}$-interpretation in question, which is not assured by the definition of $\mathcal{K}$-interpretations and hence defined explicitly.

Definition 2.11. A $\mathcal{K}$-interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{K}$ is said to be *model-defining*, if for each literal $L \in \mathrm{Lit}_A(\tau)$ exactly one of $\pi(L)$ and $\pi(\neg L)$ is assigned 0.

This requirement enables us to link classical structures to $\mathcal{K}$-interpretations in the following way. Every model-defining $\mathcal{K}$-interpretation π uniquely defines a classical model $\mathfrak{A}_\pi$ over the same universe A and vocabulary τ such that $\mathfrak{A}_\pi \models L$ if, and only if, $\pi(L) \neq 0$ for all literals $L \in \mathrm{Lit}_A(\tau)$. By this means, it can be shown that positivity of the underlying semiring ensures that the duality requirement the definition of model-defining $\mathcal{K}$-interpretations establishes for literals propagates to arbitrary formulae. More precisely, for each formula $\varphi(\bar{x})$ exactly one of $\varphi(\bar{x})$ and $\neg\varphi(\bar{x})$ is valuated with 0, which corresponds to the intention that all elements except for 0 represent a certain degree of truth.

Proposition 2.12. Let $\mathcal{K}$ be positive and π be a model-defining $\mathcal{K}$-interpretation. If π is infinite, additionally assume that $\mathcal{K}$ admits positive infinitary operations. Then, for all formulae $\varphi(x_1, \dots, x_k)$ and $\bar{a} \in A^k$ it holds that

$$\mathfrak{A}_\pi \models \varphi(\bar{a}) \text{ if, and only if, } \pi[\![\varphi(\bar{a})]\!] \neq 0.$$

2.3 The Fundamental Property

A crucial observation established in [GT17a] is given by the compatibility of semiring semantics with *semiring homomorphisms*, which map elements from one semiring into another while respecting the semiring operations as well as the identity elements as follows.

Definition 2.13. Given two semirings $(K, +^{\mathcal{K}}, \cdot^{\mathcal{K}}, 0^{\mathcal{K}}, 1^{\mathcal{K}})$ and $(L, +^{\mathcal{L}}, \cdot^{\mathcal{L}}, 0^{\mathcal{K}}, 1^{\mathcal{L}})$, a *semiring homomorphism* is a function $h\colon \mathcal{K} \to \mathcal{L}$ such that

(1) $h(0^{\mathcal{K}}) = 0^{\mathcal{L}}$ and $h(1^{\mathcal{K}}) = 1^{\mathcal{L}}$,

(2) $h(k_1 +^{\mathcal{K}} k_2) = h(k_1) +^{\mathcal{L}} h(k_2)$ and

(3) $h(k_1 \cdot^{\mathcal{K}} k_2) = h(k_1) \cdot^{\mathcal{L}} h(k_2)$.

We say that h is *compatible with the infinitary operations* admitted by $\mathcal{K}$, where appropriate, if for all families $(k_i)_{i\in I} \in K^I$ over arbitrary index sets I it holds that

(1) $h(\sum\limits_{i\in I} k_i) = \sum\limits_{i\in I} h(k_i)$ and

(2) $h(\prod\limits_{i\in I} k_i) = \prod\limits_{i\in I} h(k_i)$.

On the one hand, this central property, which is referred to as the *fundamental property*, simplifies the evaluation of a formula in multiple different semirings via polynomials as explained within the subsequent section. But it also turns out to be an essential result in the context of proving elementary equivalence.

Theorem 2.14 (Fundamental Property)**.** Let $\mathcal{K}$ and $\mathcal{L}$ be semirings, $h\colon \mathcal{K} \to \mathcal{L}$ a semiring homomorphism and $\pi\colon \operatorname{Lit}_A(\tau) \to \mathcal{K}$ a $\mathcal{K}$-interpretation. If A is infinite, further assume that $\mathcal{K}$ admits infinitary operations h is compatible with. Then, $(h \circ \pi)$ is an $\mathcal{L}$-interpretation and it holds that

$$h(\pi\llbracket\varphi(a_1,\ldots,a_n)\rrbracket) = (h\circ\pi)\llbracket\varphi(a_1,\ldots,a_n)\rrbracket$$

for all formulae $\varphi(x_1,\ldots,x_n) \in \mathrm{FO}(\tau)$ and instantiations $a_1,\ldots,a_n \in A$.

$$\begin{tikzcd}
 & \pi\llbracket\varphi(\bar{a})\rrbracket \arrow[dr, bend left, "h"] & \\
\varphi(\bar{x}) \arrow[ur, bend left, "\pi"] \arrow[rr, "(h\circ\pi)"] & & (h\circ\pi)\llbracket\varphi(\bar{a})\rrbracket
\end{tikzcd}$$

Proof. Firstly, note that the only requirement on $h\circ\pi$ for being an $\mathcal{L}$-interpretation is that all τ-literals over A are mapped to elements from $\mathcal{L}$ which is clearly fulfilled. The equation can be shown for all formulae $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ by induction over the structure of $\varphi(\bar{x})$. Thereby, we assume that $\varphi(\bar{x})$ is given in negation normal form and thus omit negations outside of the base cases.

Case 1 ((in)equalities). For all $\varphi(\bar{x}) = x_{i_1} \circ x_{i_2}$ with $\circ \in \{=,\neq\}$ and $1 \leq i_1, i_2 \leq n$ the equality follows from the preservation of neutral elements by h as follows.

$$\begin{aligned}
h(\pi\llbracket x_{i_1} = x_{i_2}\rrbracket) &= \left\{\begin{array}{ll} h(1^K), & a_{i_1} = a_{i_2} \\ h(0^K), & a_{i_1} \neq a_{i_2} \end{array}\right\} \\
&= \left\{\begin{array}{ll} 1^L, & a_{i_1} = a_{i_2} \\ 0^L, & a_{i_1} \neq a_{i_2} \end{array}\right\} = (h\circ\pi)\llbracket x_{i_1} = x_{i_2}\rrbracket \text{ and}
\end{aligned}$$

$$
\begin{aligned}
h(\pi[\![x_{i_1} \neq x_{i_2}]\!]) &= \left\{ \begin{array}{ll} h(0^K), & a_{i_1} = a_{i_2} \\ h(1^K), & a_{i_1} \neq a_{i_2} \end{array} \right\} \\
&= \left\{ \begin{array}{ll} 0^L, & a_{i_1} = a_{i_2} \\ 1^L, & a_{i_1} \neq a_{i_2} \end{array} \right\} = (h \circ \pi)[\![x_{i_1} \neq x_{i_2}]\!].
\end{aligned}
$$

Case 2 (τ-literals). If $\varphi(\bar{x}) \in \{Rx_{i_1} \dots x_{i_r}, \neg Rx_{i_1} \dots x_{i_r}\}$ with $R \in \tau$, $\mathrm{arity}(R) = r$ and $1 \leq x_{i_1}, \dots, x_{i_r} \leq n$, it holds that $\varphi(\bar{a}) \in \mathrm{Lit}_A(\tau)$, thus the equality is true by definition.

Case 3 (disjunction, conjunction). For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$, where $\circ \in \{\vee, \wedge\}$, applying the induction hypothesis yields that $h(\pi_A[\![\psi(\bar{a})]\!]) = (h \circ \pi_A)[\![\psi(\bar{a})]\!]$ and $h(\pi_A[\![\vartheta(\bar{a})]\!]) = (h \circ \pi_A)[\![\vartheta(\bar{a})]\!]$. This implies that

$$
\begin{aligned}
h(\pi_A[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!]) &= h(\pi_A[\![\psi(\bar{a})]\!] + \pi_A[\![\vartheta(\bar{a})]\!]) \\
&= h(\pi_A[\![\psi(\bar{a})]\!]) + h(\pi_A[\![\vartheta(\bar{a})]\!]) \\
&= (h \circ \pi_A)[\![\psi(\bar{a})]\!] + (h \circ \pi_A)[\![\vartheta(\bar{a})]\!] = (h \circ \pi_A)[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] \text{ and} \\
h(\pi_A[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!]) &= h(\pi_A[\![\psi(\bar{a})]\!] \cdot \pi_A[\![\vartheta(\bar{a})]\!]) \\
&= h(\pi_A[\![\psi(\bar{a})]\!]) \cdot h(\pi_A[\![\vartheta(\bar{a})]\!]) \\
&= (h \circ \pi_A)[\![\psi(\bar{a})]\!] \cdot (h \circ \pi_A)[\![\vartheta(\bar{a})]\!] = (h \circ \pi_A)[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!].
\end{aligned}
$$

Case 4 (quantifiers). For $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$, the induction hypothesis implies that $h(\pi_A[\![\psi(\bar{a}, a)]\!]) = (h \circ \pi_A)[\![\psi(\bar{a}, a)]\!]$ for all $a \in A$. It can be inferred that

$$
\begin{aligned}
h(\pi_A[\![\exists x\psi(\bar{a}, x)]\!]) &= h(\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!]) \\
&\overset{(*)}{=} \sum_{a \in A} h(\pi_A[\![\psi(\bar{a}, a)]\!]) \\
&= \sum_{a \in A} (h \circ \pi_A)[\![\psi(\bar{a}, a)]\!] = (h \circ \pi_A)[\![\exists x\psi(\bar{a}, x)]\!] \text{ and} \\
h(\pi_A[\![\forall x\psi(\bar{a}, x)]\!]) &= h(\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!]) \\
&\overset{(*)}{=} \prod_{a \in A} h(\pi_A[\![\psi(\bar{a}, a)]\!]) \\
&= \prod_{a \in A} (h \circ \pi_A)[\![\psi(\bar{a}, a)]\!] = (h \circ \pi_A)[\![\forall x\psi(\bar{a}, x)]\!]
\end{aligned}
$$

which completes the induction. □

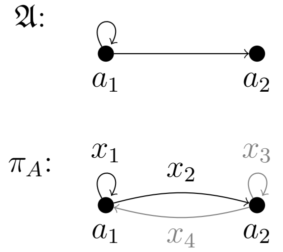
Notice that in order to deduce the steps which are marked by $(*)$, the compatibility of h with finite summation and multiplication in $\mathcal{K}$, which is already implied by h

being a semiring homomorphism, suffices if A is finite, whereas the compatibility must be lifted to the infinitary operations in case of infinite A. The remaining reasoning does without any assumptions on the infinitary operations, which is why the additional premise that h is compatible with the infinitary operations can be omitted if A is finite.

2.4 Provenance Semirings

We already observed that by means of annotating atomic facts with elements from an appropriate application semiring, the evaluation of a formula does not only determine *whether* a formula is satisfies but also *how*, e.g., a required access level or cost is provided in addition.

Beyond that, semiring semantics supplies a general framework to understand *why* a sentence is satisfies by a classical structure, so to conduct a provenance analysis on the model checking problem. This can be realized by fixing a finite set X of abstract provenance tokens and interpreting formulae by elements from an appropriate provenance semiring consisting of polynomials which are generated by the set X. As a starting point, we consider the semiring $\mathbb{N}[X]$ of multivariate polynomials with indeterminates from X and coefficients from $\mathbb{N}$ and the $\{E\}$-structure $\mathfrak{A}$ depicted below, where E is a binary relation symbol. We transform $\mathfrak{A}$ into an $\mathbb{N}[X]$-interpretation with $X = \{x_1, x_2, x_3, x_4\}$ being the set of provenance tokens used to label the $\{E\}$-literals over $\{a_1, a_2\}$ satisfied by $\mathfrak{A}$. In the resulting $\mathbb{N}[X]$-interpretation π_A, which is illustrated in the drawing, the black edges correspond to literals Ea_ia_j, while the gray edges mark literals of the form $\neg Ea_ia_j$, so, for instance, $\pi_A(\neg Ea_2a_2) = x_3$. All remaining $\{E\}$-literals over $\{a_1, a_2\}$ are valuated with 0.



$$\pi_A[\![\underbrace{\exists x \exists y \exists z(Exy \wedge (Eyy \vee Eyz))}_{\varphi}]\!] = 3x_1^2 + x_1x_2$$

The resulting valuation of the sentence φ is composed as follows. There are three different evaluation strategies using the fact that a_1 has a loop twice. One can assign a_1 to x, y as well as z and use either the subformula Eyy or Eyz to show that $\mathfrak{A}$ is a model of φ. Alternatively, z can be assigned with a_2 instead of a_1

if Eyy is used. When using this assignment in combination with the subformula Eyz, it suffices to make use of the edges corresponding to x_1 and x_2 each once. So in general, each monomial $m \cdot x_1^{e_1} \cdots x_k^{e_k}$ which is contained in the resulting valuation of a formula φ indicates that there are exactly m evaluation strategies to determine that $\mathfrak{A} \models \varphi$ which rely on the literals labeled by $x_1, \ldots x_k$ and use any literal labeled with x_i exactly e_i times.

Moreover, due to the universality of polynomial semirings, the transformation from $\mathfrak{A}$ into π_A also simplifies the evaluation of a formula in multiple different $\mathcal{K}$-interpretations which might relate to different semirings but share the same set of true literals with $\mathfrak{A}$. Instead of evaluating the formula several times, applying the fundamental property allows us to save computational resources by evaluating the formula in π_A and substituting the variables by the corresponding valuations afterwards.

Proposition 2.15 (Universal Property)**.** For each commutative semiring $\mathcal{K}$, every assignment $\alpha\colon X \to \mathcal{K}$ induces a unique homomorphism $h_\alpha\colon \mathbb{N}[X] \to \mathcal{K}$ such that $h_\alpha(x) = \alpha(x)$ for all $x \in X$.

Interpreting formulae by polynomials is also interesting for conducting a *reverse provenance analysis*, which is not based on a certain classical structure but aims at understanding how a model of a certain formula has to look like. Classical polynomials can be used to label any unnegated τ-literal over a fixed finite universe by a unique variable such that the evaluation of a positive formula φ explains which atomic properties have to be satisfied such that φ is satisfied. This approach can be extended to full first-order logic by incorporating a second set of indeterminates representing negated τ-literals. The elements of the resulting provenance semirings are referred to as *dual-indeterminate polynomials*, which we will not focus on within this thesis. Instead, we consider classical polynomial semirings such as $\mathbb{N}[X]$ as well as less informative provenance semirings whose elements correspond to congruence classes on $\mathbb{N}[X]$ which are induced by the semiring properties introduced earlier in this chapter and hold the universal property for the corresponding classes of semirings.

- The semiring $\mathbb{B}[X]$ is induced by the smallest congruence on $\mathbb{N}[X]$ such that $x + x \sim x$ for all $x \in X$ and has the univeral property for the class of idempotent semirings. The polynomials in $\mathbb{B}[X]$ arise from those in $\mathbb{N}[X]$ by dropping coefficients.
- $\mathbb{W}[X]$ emerges from $\mathbb{B}[X]$ via the congruence generated by $x \cdot x \sim x$, hence exponents are collapsed as well. It holds the universal property for the class of fully idempotent semirings.

- The polynomials in $\mathbb{S}[X]$ only contain monomials which are maximal with respect to the *absorption order.* We say that m_1 absorbs m_2, denoted as $m_1 \succcurlyeq m_2$, if $m_2 = m \cdot m_1$ for some monomial m, so if m_1 has smaller exponents than m_2. $\mathbb{S}[X]$ is the quotient semiring of $\mathbb{N}[X]$ (or, equivalently, of $\mathbb{B}[X]$) regarding the congruence generated by $f + fg \sim f$ for $f, g \in \mathbb{N}[X]$ and is universal for the class of absorptive semirings.

- By collapsing exponents in $\mathbb{S}[X]$ via the congruence generated by $x \cdot x \sim x$ for all $x \in X$ we obtain the semiring $\text{PosBool}[X]$, which is universal for the class of fully idempotent and absorptive semirings, corresponding to distributive lattices.

Finite addition and multiplication in the above-mentioned semirings arises from the usual addition and multiplication of polynomials by dropping coefficients or exponents afterwards, or only keeping the maximal monomials with regard to the absorption order as described. By contrast, it remains to check whether suitable infinitary operations are admitted as well. Firstly, consider $\mathbb{N}[X]$ and recall the finite addition and multiplication. As the sum of polynomials p_1 and p_2 is obtained by summing up the coefficients of the monomials which only differ from each other by their coefficient, it suffices to define an infinitary summation on the coefficients in order to obtain an infinitary addition of polynomials. However, we have already observed that $\mathbb{N}$ does not admit infinitary summation, so neither does $\mathbb{N}[X]$. In order to be able to evaluate first-order formulae on interpretations which label the τ-literals over an infinite universe by polynomials as well, we extend $\mathbb{N}[X]$ such that adequate infinitary operations are admitted. For this purpose, we incorporate an additional coefficient ∞ with $n + \infty = \infty$ for $n \in \mathbb{N}$ and $n \cdot \infty = \infty$ for all $n \in \mathbb{N}_{>0}$ and define infinitary sums as the supremum of the finite subsums. Whereas this modification enables the definition of sums such as $\sum_{i \in \mathbb{N}} x$ appropriately, we must additionally drop the finiteness of the polynomials and consider *formal power series* in order to cope with sums such as $\sum_{i \in \mathbb{N}} x^i$. The resulting semiring, which extends $\mathbb{N}[X]$ and admits infinitary summation, is denoted as $\mathbb{N}^\infty[\![X]\!]$. In order to check whether $\mathbb{N}^\infty[\![X]\!]$ admits appropriate infinitary multiplication as well, recall that, due to distributivity, the product of two polynomials can be calculated as

$$p_1 \cdot p_2 = \sum_{\substack{m_1 \in p_1, \\ m_2 \in p_2}} m_1 m_2,$$

where $m_i \in p_i$ is supposed to denote that the monomial m_i occurs in p_i. Thus, the natural generalization of finite multiplication to arbitrary families $(p_i)_{i \in I}$ of polynomials is given by

$$\prod_{i\in I} p_i = \sum_{\substack{(m_i)_{i\in I}:\\ m_i\in p_i}} \prod_{i\in I} m_i,$$

which only requires to define an adequate infinitary product on monomials. In the finite case, we simply multiply the coefficients and sum up the exponents, so it remains to specify an infinitary multiplication on the coefficients and an infinitary addition on the exponents. The former can, analogously to the summation of exponents, be defined as the supremum of the finite subproducts, whereas the latter requires to incorporate an additional exponent ∞. We denote the resulting semiring of formal power series with coefficients and exponents in $\mathbb{N} \cup \{\infty\}$ by $\mathbb{N}^{\infty}_{\infty}[\![X]\!]$. Analogously, we extend the further polynomial semirings, where required, such that infinitary operations are admitted by specifying an infinitary addition and multiplication of coefficients and an infinitary addition of the exponents.

- The semiring $\mathbb{B}_{\infty}[\![X]\!]$ extends $\mathbb{B}[X]$ by dropping the finiteness of polynomials and incorporating an additional exponent ∞. The coefficients of polynomials in $\mathbb{B}[X]$ are from $\mathbb{B}$, which admits infinitary operations as follows. Sums are evaluated to 1 if, and only if, one of the summands is 1 and products to 0 if, and only if, one of the factors is 0. The summation of exponents is defined as for $\mathbb{N}^{\infty}_{\infty}[\![X]\!]$.
- For the purpose of extending $\mathbb{S}[X]$, we make use of the semiring $\mathbb{S}_{\infty}[X]$ which only differs from $\mathbb{S}[X]$ by an additional exponent ∞. Since every antichain of monomials with respect to the absorption order is finite, infinite sums of polynomials in $\mathbb{S}_{\infty}[X]$ always result in (finite) polynomials. Hence, due to absorption, it is not necessary to move from polynomials to formal power series in order to incorporate infinitary operations. More detailed information on this can be found in [DGNT21].
- Both coefficients and exponents in polynomials from $\mathbb{W}[X]$ and $\mathrm{PosBool}[X]$ can be thought of as elements from $\mathbb{B}$. Since $\mathbb{B}$ admits infinitary operations as described, $\mathbb{W}[X]$ and $\mathrm{PosBool}[X]$ do as well.

Consequently, we will only consider finite $\mathbb{N}[X]$-, $\mathbb{B}[X]$- and $\mathbb{S}[X]$-interpretations and move to the semirings $\mathbb{N}^{\infty}_{\infty}[\![X]\!]$, $\mathbb{B}_{\infty}[\![X]\!]$ and $\mathbb{S}_{\infty}[X]$, respectively, whenever infinite universes shall be taken into account. By contrast, we make no demands on the universe of $\mathbb{W}[X]$- and $\mathrm{PosBool}[X]$-interpretations. Finally, the considered semirings of polynomials and formal power series and the way they emerge from each other can be summarized as follows.

$$\begin{tikzcd}
& \mathbb{N}[X] \arrow[d, "\text{drop coef.}"'] \arrow[rr, "\text{add } \infty \text{ as exp. and coef.}", "\text{drop finiteness}"'] & & \mathbb{N}^{\infty}_{\infty}[\![X]\!] \\
& \mathbb{B}[X] \arrow[dl, "\text{drop exp.}"'] \arrow[dr, "\text{absorb}"] \arrow[rr, "\text{add } \infty \text{ as exp.}", "\text{drop finiteness}"'] & & \mathbb{B}_{\infty}[\![X]\!] \\
\mathbb{W}[X] \arrow[dr, "\text{absorb}"'] & & \mathbb{S}[X] \arrow[dl, "\text{drop exp.}"] \arrow[r, "\text{add } \infty \text{ as exp.}"] & \mathbb{S}_{\infty}[X] \\
& \mathrm{PosBool}[X] & &
\end{tikzcd}$$

Figure 2.2: A hierarchy of provenance semirings, adapted from [GT17b], and their extensions admitting infinitary operations.

2.5 Equivalence and Isomorphism

Equivalence and isomorphism constitute the central notions examined in this thesis, as the classical Ehrenfeucht-Fraïssé game interlinks them with regard to standard first-order logic. So, in order to analyze the transferability to semiring semantics, it remains to adjust these definitions. The most natural generalization of elementary equivalence, as introduced in [GM21], demands that two given $\mathcal{K}$-interpretations map all formulae to the same semiring element, respectively, instead of satisfying the same formulae. Just as for classical semantics, we specify this definition by means of the quantifier rank.

Definition 2.16. Let $\pi_A\colon \mathrm{Lit}_A(\tau) \to \mathcal{K}$ and $\pi_B\colon \mathrm{Lit}_B(\tau) \to \mathcal{K}$ be $\mathcal{K}$-interpretations and $\bar{a} \in A^n$, $\bar{b} \in B^n$ be tuples of elements. $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$ are

- *m-equivalent*, denoted by $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$, if $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ holds for all $\varphi(x_1, \dots, x_n) \in \mathrm{FO}(\tau)$ with $\mathrm{qr}(\varphi(\bar{x})) \leq m$.
- *elementarily equivalent*, denoted by $(\pi_A, \bar{a}) \equiv (\pi_B, \bar{b})$, if $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$ are m-equivalent for all $m \in \mathbb{N}$.

In an analogous way, we extend the definition of isomorphisms based on the intuition that isomorphic $\mathcal{K}$-interpretations coincide with each other up to the naming of their elements. In order to give a formal definition, we write $\mathrm{Lit}_n(\tau)$ to denote the set of τ-literals $R\bar{x}$ and $\neg R\bar{x}$ where $R \in \tau$ and $\bar{x}$ is a tuple of variables from $\{x_1, \dots, x_n\}$ which has length $\mathrm{arity}(R)$.

Definition 2.17. For $\mathcal{K}$-interpretations $\pi_A\colon \mathrm{Lit}_A(\tau) \to \mathcal{K}$, $\pi_B\colon \mathrm{Lit}_B(\tau) \to \mathcal{K}$ and elements $\bar{a} = (a_1, \ldots, a_n) \in A^n$ and $\bar{b} = (b_1, \ldots, b_n) \in B^n$ we write $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ and refer to them as *isomorphic*, if there is a bijection $\sigma\colon A \to B$ such that

(1) $\sigma(a_i) = b_i$ for all $1 \leq i \leq n$ and

(2) $\pi_A(L(\bar{a})) = \pi_B(L(\bar{b}))$ for all $L(x_1, \ldots, x_n) \in \mathrm{Lit}_n(\tau)$,

If both conditions are fulfilled, the mapping σ is said to be an *isomorphism* between $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$.

Just as for classical semantics, first-order logic with semiring semantics cannot distinguish between isomorphic $\mathcal{K}$-interpretations, which is formalized by the following isomorphism lemma.

Lemma 2.18 (Isomorphism Lemma)**.** For $\mathcal{K}$-interpretations π_A, π_B and $\bar{a} \in A^n$, $\bar{b} \in B^n$ it holds that $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ implies $(\pi_A, \bar{a}) \equiv (\pi_B, \bar{b})$.

Proof. We show that $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ implies $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ for all $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\bar{x} = (x_1, \ldots, x_n)$, $\bar{a} = (a_1, \ldots, a_n) \in A^n$, and $\bar{b} = (b_1, \ldots, b_n) \in B^n$ by induction over the structure of $\varphi(\bar{x})$. As before, $\varphi(\bar{x})$ is assumed to be available in negation normal form.

Case 1 ((in)equalities). Since $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ by assumption, there must be a bijection σ such that $\sigma(a_i) = b_i$ for all $1 \leq i \leq n$. This immediately implies $a_{i_1} = a_{i_2}$ if, and only if, $b_{i_1} = b_{i_2}$, thus $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ for all $\varphi(\bar{x}) = x_{i_1} \circ x_{i_2}$ with $\circ \in \{=, \neq\}$ and $1 \leq i_1, i_2 \leq n$.

Case 2 (τ-literals). For $\varphi(\bar{x}) \in \{Rx_{i_1} \ldots x_{i_r}, \neg Rx_{i_1} \ldots x_{i_r}\}$, where $R \in \tau$, $\mathrm{arity}(R) = r$ and $1 \leq x_{i_1}, \ldots, x_{i_r} \leq n$, the existence of an isomorphism σ between $(\pi_A, \bar{a})$ and $(\pi_A, \bar{a})$ implies that each a_i can be mapped to b_i such that all literals with corresponding instantiations according to σ are valuated equally in π_A and π_B. Hence, $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ must be true.

Case 3 (disjunction, conjunction). For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$, where $\circ \in \{\vee, \wedge\}$, applying the induction hypothesis yields that $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ implies $\pi_A[\![\psi(\bar{a})]\!] = \pi_B[\![\psi(\bar{b})]\!]$ and $\pi_A[\![\vartheta(\bar{a})]\!] = \pi_B[\![\vartheta(\bar{b})]\!]$. Hence,

$$\begin{aligned}
\pi_A[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] + \pi_A[\![\vartheta(\bar{a})]\!] \\
&= \pi_B[\![\psi(\bar{b})]\!] + \pi_B[\![\vartheta(\bar{b})]\!] = \pi_B[\![\psi(\bar{b}) \vee \vartheta(\bar{b})]\!] \text{ and} \\
\pi_A[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] \cdot \pi_A[\![\vartheta(\bar{a})]\!] \\
&= \pi_B[\![\psi(\bar{b})]\!] \cdot \pi_B[\![\vartheta(\bar{b})]\!] \ = \pi_B[\![\psi(\bar{b}) \wedge \vartheta(\bar{b})]\!]
\end{aligned}$$

must hold as well.

Case 4 (quantifiers). For $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$, let σ be an isomorphism between $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$. Clearly, $(\pi_A, \bar{a}, a) \cong (\pi_B, \bar{b}, \sigma(a))$ holds as well for all $a \in A$. By induction hypothesis this implies $\pi_A[\![\psi(\bar{a}, a)]\!] = \pi_B[\![\psi(\bar{b}, \sigma(a))]\!]$ for all $a \in A$. It can be inferred that

$$\begin{aligned}
\pi_A[\![\exists x\psi(\bar{a}, x)]\!] &= \sum_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!] \\
&= \sum_{a\in A} \pi_B[\![\psi(\bar{b}, \sigma(a))]\!] \\
&\overset{(*)}{=} \sum_{b\in B} \pi_B[\![\psi(\bar{b}, b)]\!] \qquad = \pi_B[\![\exists x\psi(\bar{b}, x)]\!] \text{ and} \\
\pi_A[\![\forall x\psi(\bar{a}, x)]\!] &= \prod_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!] \\
&= \prod_{a\in A} \pi_B[\![\psi(\bar{b}, \sigma(a))]\!] \\
&\overset{(*)}{=} \prod_{b\in B} \pi_B[\![\psi(\bar{b}, b)]\!] \qquad = \pi_B[\![\forall x\psi(\bar{b}, x)]\!].
\end{aligned}$$

Note that the steps which are marked by $(*)$ rely on invariance under bijections in case A and B are infinite. Therefore, we can overall conclude that $(\pi_A, \bar{a}) \cong (\pi_B, \bar{b})$ implies $(\pi_A, \bar{a}) \equiv (\pi_B, \bar{b})$. $\square$

In classical semantics, the converse direction holds as well when considering only finite structures, so in the Boolean case, finite structures can be defined up to isomorphism. As shown in [GM21], this is only true for particular semirings such as the Viterbi or natural semiring when moving from classical to semiring semantics.

Following the aforementioned definition, for any two $\mathcal{K}$-interpretations which are not equivalent, there must be a formula valuated differently in the $\mathcal{K}$-interpretations, which we refer to as *separating*. When fixing a separating formula the outermost Boolean combination can be omitted, as justified by the following lemma.

Lemma 2.19. If $(\pi_A, \bar{a}) \not\equiv_m (\pi_B, \bar{b})$ for some $m \in \mathbb{N}$, then there is a separating formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\mathrm{qr}(\varphi) \leq m$ which is a literal or has the form

$$\varphi(\bar{x}) = \exists x\psi(\bar{x}, x) \text{ or } \varphi(\bar{x}) = \forall x\psi(\bar{x}, x)$$

for some $\psi(\bar{x}, x) \in \mathrm{FO}(\tau)$.

Proof. Fix some formula $\vartheta(\bar{x}) \in \mathrm{FO}(\tau)$ with quantifier rank at most m which separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. $\vartheta(\bar{x})$ is a positive Boolean combination of formulae

$\varphi_i(\bar{x})$ which are literals or of the form $\varphi_i(\bar{x}) = \exists x \psi_i(\bar{x})$ or $\varphi_i(\bar{x}) = \forall x \psi_i(\bar{x})$. We claim that there must be some subformula $\varphi_i(\bar{x})$ already separating $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$.

Assume that $\vartheta(\bar{x}) = \vartheta_1(\bar{x}) \circ \vartheta_2(\bar{x})$, where $\circ \in \{\vee, \wedge\}$. By assumption it holds that

$$\pi_A[\![\vartheta(\bar{a})]\!] = \pi_A[\![\vartheta_1(\bar{a})]\!] \star \pi_A[\![\vartheta_2(\bar{a})]\!] \neq \pi_B[\![\vartheta_1(\bar{b})]\!] \star \pi_B[\![\vartheta_2(\bar{b})]\!] = \pi_B[\![\vartheta(\bar{b})]\!],$$

where $\star \in \{+, \cdot\}$ correspondingly. This implies

$$\pi_A[\![\vartheta_1(\bar{a})]\!] \neq \pi_B[\![\vartheta_1(\bar{b})]\!] \text{ or } \pi_A[\![\vartheta_2(\bar{a})]\!] \neq \pi_B[\![\vartheta_2(\bar{b})]\!],$$

thus the claim follows by induction. □

We have already established that the existence of an isomorphism provides a sufficient condition for elementary equivalence. However, the isomorphism lemma does not give actual insights about the expressive power of first-order logic with semiring semantics, as the separability of isomorphic structures by a logical formalism is generally undesired. While the Ehrenfeucht-Fraïssé method provides a crucial proof technique for m-equivalence between classical structures, it is not (yet) available for semiring semantics, which poses the question how m-equivalence of $\mathcal{K}$-interpretations can be proven otherwise. Aiming at this question, a method was derived in [Mrk20], which relies on the fundamental property and allows us to reduce the problem of proving m-equivalence to another semiring by decomposing the semiring in question by means of homomorphisms.

Definition 2.20. Given semirings $\mathcal{K}$ and $\mathcal{L}$, a set H of homomorphisms from $\mathcal{K}$ to $\mathcal{L}$ is called *separating*, if for all $k, k' \in K$ with $k \neq k'$ there is some $h \in H$ with $h(k) \neq h(k')$.

Constructing sets of homomorphisms which separate any two semiring elements is based on the following idea. For two given $\mathcal{K}$-interpretations π_A and π_B which are separable by some sentence φ, we can think of the valuations $k \neq k'$ of φ in π_A and π_B, respectively, as witnesses for the separability of π_A and π_B. Further, whenever there is a homomorphism h such that $h(k) \neq h(k')$ and $(h \circ \pi_A) \equiv (h \circ \pi_B)$, we can exclude the pair (k, k') as a candidate for witnessing $\pi_A \not\equiv \pi_B$ due to the fundamental property. Consequently, a separating set H contains enough homomorphisms such that elementary equivalence of the $\mathcal{K}$-interpretations in question can be inferred from elementary equivalence under all homomorphisms in H.

Lemma 2.21. Let $\mathcal{K}$ and $\mathcal{L}$ be semirings and H a separating set of homomorphisms from $\mathcal{K}$ to $\mathcal{L}$. Further, let π_A, π_B be $\mathcal{K}$-interpretations, $\bar{a} \in A^n$ and $\bar{b} \in B^n$ and suppose that each $h \in H$ is compatible with the infinitary operations in $\mathcal{K}$ if π_A or π_B is infinite. It holds that $(h \circ \pi_A, \bar{a}) \equiv_m (h \circ \pi_B, \bar{b})$ for all $h \in H$ if, and only if, $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$.

Proof. ($\Rightarrow$) In order to show the contraposition, assume that $(\pi_A, \bar{a}) \not\equiv_m (\pi_B, \bar{b})$ and let $\varphi(\bar{x})$ be a separating formula of quantifier rank at most m. Since H is separating, there must be some $h \in H$ such that $h(\pi_A[\![\varphi(\bar{a})]\!]) \neq h(\pi_B[\![\varphi(\bar{b})]\!])$. Applying the fundamental property yields $(h \circ \pi_A)[\![\varphi(\bar{a})]\!] \neq (h \circ \pi_B)[\![\varphi(\bar{b})]\!])$, hence $\varphi(\bar{x})$ separates $(h \circ \pi_A, \bar{a})$ and $(h \circ \pi_B, \bar{b})$ and we can infer $(h \circ \pi_A, \bar{a}) \not\equiv_m (h \circ \pi_B, \bar{b})$.

($\Leftarrow$) Again, we prove the contraposition and suppose that there is some $h \in H$ such that $(h \circ \pi_A, \bar{a}) \not\equiv_m (h \circ \pi_B, \bar{b})$. Let $\varphi(\bar{x})$ be a separating formula with $\mathrm{qr}(\varphi(\bar{x})) \leq m$. We have that $h(\pi_A[\![\varphi(\bar{a})]\!]) = (h \circ \pi_A)[\![\varphi(\bar{a})]\!] \neq (h \circ \pi_B)[\![\varphi(\bar{b})]\!] = h(\pi_B[\![\varphi(\bar{b})]\!])$ by the fundamental theorem. Therefore $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ cannot be true and it holds that $(\pi_A, \bar{a}) \not\equiv_m (\pi_B, \bar{b})$. $\square$

Chapter 3

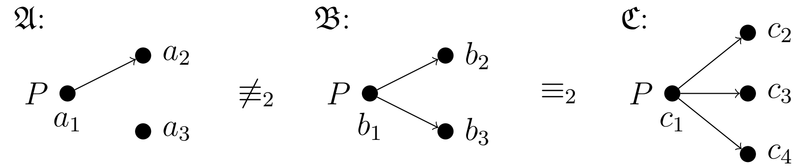
Transferability of the Classical Games

In classical model theory, games characterizing elementary equivalence in a certain logic play a central role, as they provide a convenient method to analyze the expressive power of the logical formalism, which is measured by its ability to distinguish between structures. Most prominently, the Ehrenfeucht-Fraïssé game captures m-equivalence in classical first-order logic. It traces back to an algebraic notion of indistinguishability by first-order formulae proposed by Fraïssé [Fra55], which was then reformulated in game-theoretic terminology by Ehrenfeucht [Ehr61]. The main idea behind these kinds of games is to avoid keeping track of all formulae that may potentially separate a given pair of structures but to consider a game on the structures which is played by the two players *Spoiler* and *Duplicator*. Whereas Spoiler claims that the structures can be separated, it is Duplicator's objective to show their indistinguishability such that equivalence can be inferred if, and only if, Duplicator has a winning strategy in the game.

In order to simulate the use of quantifiers in a potentially separating first-order formula, in each turn of the Ehrenfeucht-Fraïssé game, Spoiler chooses an element in one of the structures and Duplicator has to respond with some element in the other structure. If there is a separating formula of the form $\varphi = \exists x \psi(x)$ which is satisfied by exactly one of the structures, say A, then there is some element $a \in A$ witnessing $\mathfrak{A} \models \varphi$. This element can be chosen by Spoiler, so he can challenge Duplicator to find a duplicate in B witnessing $\mathfrak{B} \models \varphi$, which is not possible if φ indeed is a separating formula. In this manner, drawing elements in each turn eliminates the quantifiers which might be used in a separating formula. Thus, it suffices to compare the structures and elements which are chosen during the play on the atomic level. More precisely, Duplicator wins a play, if and only if, the selected elements constitute a *local isomorphism*, that is, an isomorphism between the induced substructures.

Duplicator wins the m-turn Ehrenfeucht-Fraïssé game on $\mathfrak{A}$ and $\mathfrak{B}$ if, and only if, $\mathfrak{A}$ and $\mathfrak{B}$ are m-equivalent.

Example 3.1. Consider the following structures over the vocabulary consisting of a unary relation symbol P and a binary relation symbol E denoted by edges.



The game played on $\mathfrak{A}$ and $\mathfrak{B}$ for two turns is won by Spoiler. He can first choose $b_1 \in B$, if Duplicator answers with a_2 or a_3, she loses, because $b_1 \in P^{\mathfrak{B}}$ but $a_2, a_3 \notin P^{\mathfrak{A}}$. Otherwise, Duplicator chooses a_1 and Spoiler can continue to pick $a_3 \in A$. Duplicator would have to answer with some $b \in B$ such that $b \neq b_1$, because $a_3 \neq a_1$ and $(b_1, b) \notin E^{\mathfrak{B}}$, as $(a_1, a_3) \notin E^{\mathfrak{A}}$. This is not possible, hence Spoiler wins. Indeed, the formula $\varphi := \exists x(Px \wedge \forall y(x = y \vee Exy))$, which has quantifier rank 2, separates $\mathfrak{A}$ from $\mathfrak{B}$.

In contrast, two turns played on $\mathfrak{B}$ and $\mathfrak{C}$ are won by Duplicator. Duplicator can answer b_1 with c_1, any element from $\{b_2, b_3\}$ with some arbitrary element from $\{c_2, c_3, c_4\}$ and vice versa in the first turn. In the second turn, she can proceed analogously but has to make sure that the equalities regarding the elements that were chosen in the first turn are respected. Since both $\{b_2, b_3\}$ and $\{c_2, c_3, c_4\}$ contain at least 2 elements, this is possible and Duplicator can uphold the winning condition for two turns.

A central observation which causes the game rules to be appropriate to capture m-equivalence with regard to classical semantics is that structures $\mathfrak{A}$ and $\mathfrak{B}$ can be separated by $\exists x\psi(x)$ or $\forall x\psi(x)$ if, and only if, there is some $a \in A$ (or $b \in B$) such that for all $b \in B$ (or $a \in A$, respectively) the formula $\psi(x)$ separates $(\mathfrak{A}, a)$ from $(\mathfrak{B}, b)$. The equivalence holds in classical first-order logic, since for any structure $\mathfrak{A}$, the semantics of both $\exists x\psi(x)$ and $\forall x\psi(x)$ is uniquely determined by the existence of an element $a \in A$ such that $\mathfrak{A} \models \varphi(a)$ or $\mathfrak{A} \not\models \varphi(a)$, respectively. As the following counterexamples, which rely on the vocabulary τ consisting of a single unary relation symbol R, illustrate, neither of the implications translates to semiring semantics.

$(\mathbb{N}, +, \cdot, 0, 1)$

π^1_A:

A	R	$\neg R$
a_1	1	0
a_2	1	0
a_3	2	0

π^1_B:

B	R	$\neg R$
b_1	1	0
b_2	2	0
b_3	2	0

$$\pi^1_A[\![\exists x Rx]\!] = 4 \neq 5 = \pi^1_B[\![\exists x Rx]\!]$$

$(\{0, 1, 2, 3, 4\}, \max, \min, 0, 4)$

π^2_A:

A	R	$\neg R$
a_1	1	0
a_2	2	0
a_3	4	0

π^2_B:

B	R	$\neg R$
b_1	1	0
b_2	3	0
b_3	4	0

$$\pi^2_A[\![\exists x Rx]\!] = 4 = \pi^2_B[\![\exists x Rx]\!]$$
$$\pi^2_A[\![\forall x Rx]\!] = 1 = \pi^2_B[\![\forall x Rx]\!]$$

The $\mathbb{N}$-interpretations π^1_A and π^1_B can be separated by the sentence $\exists x Rx$, even though every valuation in π^1_A with regard to R can be duplicated in π^1_B and vice versa. Thus, the number of occurrences of the valuations 1 and 2 affect the valuation of $\exists x Rx$, i.e., the resulting sum in $\mathbb{N}$. On the other hand, π^2_A and π^2_B can neither be distinguished by $\exists x Rx$ nor by $\forall x Rx$, although the valuations of Ra_2 and Rb_2 are not duplicable in the respective other interpretation. So, there can be valuations whose occurrences neither change the resulting sum nor the product.

These observations provide first insights into the crucial differences between m-equivalence with regard to classical and semiring semantics and suggest that a direct adaptation of the Ehrenfeucht-Fraïssé game rules for capturing m-equivalence of $\mathcal{K}$-interpretations poses problems. This raises the following questions, which will be examined within the following chapters.

(1) How can the Ehrenfeucht-Fraïssé game be transferred to $\mathcal{K}$-interpretations?

(2) Does the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations characterize m-equivalence in semiring semantics? In which cases does it fail and why?

(3) How does the situation change when considering common variants of the classical Ehrenfeucht-Fraïssé game?

(4) What notion of equivalence is captured by the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations?

(5) How do the game rules need to be adjusted to capture m-equivalence in semiring semantics?

We begin by taking a closer look at questions (1) and (2), the following section is dedicated to.

3.1 The Ehrenfeucht-Fraïssé Game

The straightforward approach to applying the Ehrenfeucht-Fraïssé game to $\mathcal{K}$-interpretations is to make use of the same positions and to permit the same moves as in the original game. It only remains to specify the winning condition, that is, to define local isomorphisms on $\mathcal{K}$-interpretations. Analogous to classical structures, this can be realized by considering the induced subinterpretations.

Definition 3.2. Let π_A and π_B be $\mathcal{K}$-interpretations and $\bar{a} = (a_1, \dots, a_n) \in A^n$, $\bar{b} = (b_1, \dots, b_n) \in B^n$. The mapping $\sigma\colon \{a_1, \dots, a_n\} \to \{b_1, \dots, b_n\}$ defined by $\sigma\colon a_i \mapsto b_i$ for $1 \leq i \leq n$ is a *local isomorphism* between π_A and π_B if σ is an isomorphism between the induced subinterpretations $\pi_A|_{\mathrm{Lit}_{\{a_1,\dots a_n\}}(\tau)}$ and $\pi_B|_{\mathrm{Lit}_{\{b_1,\dots b_n\}}(\tau)}$.

Besides the m-turn Ehrenfeucht-Fraïssé game G_m, we also transfer its variant G to $\mathcal{K}$-interpretations. The game G is won by Duplicator if, and only if, she wins G_m for all $m \in \mathbb{N}$ and thus captures elementary equivalence with regard to classical first-order logic.

Definition 3.3. Let π_A, π_B be $\mathcal{K}$-interpretations over relational vocabulary τ with universes A, B such that $A \cap B = \varnothing$. Each play of the *Ehrenfeucht-Fraïssé game* $G_m(\pi_A, \pi_B)$ consists of m moves. In the i-th move, Spoiler chooses some element $a_i \in A$ or $b_i \in B$ and Duplicator answers with an element in the other $\mathcal{K}$-interpretation. Afterwards, the play is at position $(a_1, \dots, a_i, b_1, \dots, b_i)$ and the remaining subgame is denoted by $G_{m-i}(\pi_A, a_1, \dots, a_i, \pi_B, b_1, \dots, b_i)$. Subsequent to all m turns, elements $a_1, \dots, a_m \in A$ and $b_1, \dots, b_m \in B$ have been chosen and Duplicator wins the play if, and only if, the mapping $\sigma\colon a_i \mapsto b_i$ where $1 \leq i \leq m$ is a local isomorphism between π_A and π_B.

In the game $G(\pi_A, \pi_B)$, Spoiler chooses some $m \in \mathbb{N}$ at the beginning of each play. Afterwards, the game $G_m(\pi_A, \pi_B)$ is played.

With the game rules being adjusted to $\mathcal{K}$-interpretations, we will now analyze the transferability of the game G_m to m-equivalence under semiring semantics by answering the following questions.

(1) Soundness: Does the existence of a winning strategy for Duplicator in the game $G_m(\pi_A, \pi_B)$ imply $\pi_A \equiv_m \pi_B$?

(2) Completeness: Does $\pi_A \equiv_m \pi_B$ ensure the winning of Duplicator in the game $G_m(\pi_A, \pi_B)$?

As a reachability game which admits finite plays only, G_m is determined, which is why the completeness of G_m is equivalent to $\pi_A \not\equiv_m \pi_B$ if Spoiler wins $G_m(\pi_A, \pi_B)$.

Analogously, we examine the soundness and completeness of the game G as a proof method for elementary equivalence of $\mathcal{K}$-interpretations.

The fact that Boolean quantifiers do not allow counting is one of the central properties of classical first-order logic the classical Ehrenfeucht-Fraïssé game relies on. If structures $\mathfrak{A}$ and $\mathfrak{B}$ only differ in the precise number of instantiations such that a formula $\psi(x)$ is satisfied but both satisfy $\psi(x)$ for some instantiation of x, then $\exists x\psi(x)$ cannot separate $\mathfrak{A}$ from $\mathfrak{B}$. Generalizing this property to semiring semantics leads to the following requirement: If the valuation of $\exists x\psi(x)$ differs in $\mathcal{K}$-interpretations π_A and π_B, then this cannot be solely due to the number of elements in A and B, respectively, such that $\psi(x)$ evaluates to a certain semiring element. At the beginning of this chapter, we already observed that this property does not hold in semiring semantics, as witnessed by the $\mathbb{N}$-interpretations π_A^1 and π_B^1. The fact that $\exists xRx$ separates π_A^1 and π_B^1 results from the inequality of the sums $1 + 1 + 2 \neq 1 + 2 + 2$. However, each summand in one of the sums can be duplicated in the other sum, which is why no single summand can be made responsible for the inequality of the sums. This causes Duplicator to win the game $G_1(\pi_A, \pi_B)$, although there is a separating sentence of quantifier rank 1.

What we assume when transferring the moves of the Ehrenfeucht-Fraïssé game to semiring semantics is that unequal sums or products can always be attributed to unequal sets of summands or factors, respectively. While we observed that this is not true in general, the following lemma illustrates that this assumption is justified exactly for those semirings that are fully idempotent.

Lemma 3.4. Any semiring $\mathcal{K}$ with infinitary operations Σ and Π is fully idempotent if, and only if,

(1) $\sum\limits_{i \in I} k_i \neq \sum\limits_{j \in J} \ell_j$ implies $\{k_i \colon i \in I\} \neq \{\ell_j \colon j \in J\}$ and

(2) $\prod\limits_{i \in I} k_i \neq \prod\limits_{j \in J} \ell_j$ implies $\{k_i \colon i \in I\} \neq \{\ell_j \colon j \in J\}$

for all families $(k_i)_{i \in I}$ and $(\ell_j)_{j \in J}$ over arbitrary index sets I and J.

Proof. ($\Leftarrow$): If $\mathcal{K}$ with Σ and Π is not fully idempotent, then there must be some $k \in K$ and some index set I such that $\sum_{i \in I} k \neq k$ or $\prod_{i \in I} k \neq k$. This immediately implies that (1) or (2) is violated, since $\{k \colon i \in I\} = \{k\}$.

($\Rightarrow$): Let $\mathcal{K}$ with Σ and Π be fully idempotent. To show (1) by contraposition, let $(k_i)_{i \in I} \in K^I$ and $(\ell_j)_{j \in J} \in K^J$ be given as above and suppose that $\{k_i \colon i \in I\} = \{\ell_j \colon j \in J\}$. As addition in $\mathcal{K}$ is associative, Σ is invariant under partitions and

$\mathcal{K}$ with Σ is idempotent, it holds that

$$\sum_{i\in I} k_i = \sum_{k\in K}\sum_{\substack{i\in I:\\ k_i=k}} k = \sum_{k\in\{k_i\,:\,i\in I\}} k = \sum_{k\in\{\ell_j\,:\,j\in J\}} k = \sum_{k\in K}\sum_{\substack{j\in J:\\ \ell_j=k}} k = \sum_{j\in J}\ell_j.$$

The equality of the products (2) can be inferred analogously. $\square$

Notice that if only finite index sets I and J are considered, the infinitary operations are not required and the claim follows from the full idempotence of $\mathcal{K}$. Unlike observed for the $\mathbb{N}$-interpretations π_A^1 and π_B^1, Spoiler can always find an instantiation in $\mathcal{K}$-interpretations π_A, π_B such that the valuation of $\psi(x)$ cannot be duplicated in the respective other $\mathcal{K}$-interpretation if $\exists x\psi(x)$ or $\forall x\psi(x)$ is separating and $\mathcal{K}$ is fully idempotent. Thus, it can be shown that full idempotence constitutes an algebraic characterization of the class of semirings for which m-equivalence can be inferred based on the game G_m.

Theorem 3.5. Let $\mathcal{K}$ be a semiring (with infinitary operations Σ and Π). The existence of a winning strategy for Duplicator in $G_m(\pi_A, \pi_B)$ implies $\pi_A \equiv_m \pi_B$ for all $\mathcal{K}$-interpretations π_A and π_B and all $m \in \mathbb{N}$ if, and only if, $\mathcal{K}$ (with Σ and Π) is fully idempotent.

Proof. ($\Leftarrow$): Suppose that $\mathcal{K}$ (with Σ and Π) is fully idempotent. Based on a separating formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\pi_A[\![\varphi(\bar{a})]\!] \neq \pi_B[\![\varphi(\bar{b})]\!]$ and $\mathrm{qr}(\varphi(\bar{x})) \leq m$ where $\bar{a} \in A^n$ and $\bar{b} \in B^n$, we construct a winning strategy for Spoiler in the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by induction on the structure of $\varphi(\bar{x})$.

Case 1. If $\varphi(\bar{x})$ is a literal, then the mapping $\sigma\colon \bar{a} \mapsto \bar{b}$ cannot be a local isomorphism, which violates the winning condition. Thus, Spoiler wins the game $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$ where $\circ \in \{\vee, \wedge\}$ and $\mathrm{qr}(\varphi(\bar{x})) \leq m$, we have that

$$\begin{aligned}
\pi_A[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] + \pi_A[\![\vartheta(\bar{a})]\!] \\
&\neq \pi_B[\![\psi(\bar{b})]\!] + \pi_B[\![\vartheta(\bar{b})]\!] = \pi_B[\![\psi(\bar{b}) \vee \vartheta(\bar{b})]\!] \text{ or} \\
\pi_A[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] \cdot \pi_A[\![\vartheta(\bar{a})]\!] \\
&\neq \pi_B[\![\psi(\bar{b})]\!] \cdot \pi_B[\![\vartheta(\bar{b})]\!] \;\; = \pi_B[\![\psi(\bar{b}) \wedge \vartheta(\bar{b})]\!]
\end{aligned}$$

by assumption, which implies $\pi_A[\![\psi(\bar{a})]\!] \neq \pi_B[\![\psi(\bar{b})]\!]$ or $\pi_A[\![\vartheta(\bar{a})]\!] \neq \pi_B[\![\vartheta(\bar{b})]\!]$. It holds that $\mathrm{qr}(\psi(\bar{x})) \leq m$ and $\mathrm{qr}(\vartheta(\bar{x})) \leq m$, hence Spoiler wins $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by induction hypothesis.

Case 3. If $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ and $\mathrm{qr}(\varphi(\bar{x})) \leq m$, it holds that

$$\pi_A[\![\exists x\psi(\bar{a}, x)]\!] = \sum_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \sum_{b\in B} \pi_B[\![\psi(\bar{b}, b)]\!] = \pi_B[\![\exists x\psi(\bar{b}, x)]\!] \text{ or}$$
$$\pi_A[\![\forall x\psi(\bar{a}, x)]\!] = \prod_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \prod_{b\in B} \pi_B[\![\psi(\bar{b}, b)]\!] = \pi_B[\![\forall x\psi(\bar{b}, x)]\!].$$

By lemma 3.4, both cases imply that

$$\{\pi_A[\![\psi(\bar{a}, a)]\!] \colon a \in A\} \neq \{\pi_B[\![\psi(\bar{b}, b)]\!] \colon b \in B\}.$$

Spoiler wins the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by choosing some element $a \in A$ or $b \in B$ witnessing the inequality above. For all possible answers $b \in B$ or $a \in A$, respectively, it holds that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq \pi_B[\![\psi(\bar{b}, b)]\!]$. Applying the induction hypothesis yields that Spoiler has a winning strategy for the remaining game $G_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$, as $\mathrm{qr}(\psi(\bar{x}, x)) \leq m - 1$.

($\Rightarrow$): Suppose that summation or multiplication in $\mathcal{K}$ is not idempotent. Then, there is some $k \in K$ and some index set $i \in I$ such that

$$\sum_{i\in I} k \neq k \quad \text{or} \quad \prod_{i\in I} k \neq k.$$

Based on this element k, we construct $\mathcal{K}$-interpretations π_A^k and π_B^k with universes $A := \{a_i \colon i \in I\}$ and $B := \{b\}$ as follows.

$$\pi_A^k : \begin{array}{c||c|c} A & R & \neg R \\ \hline\hline a_i & k & 0^{\mathcal{K}} \\ \hline a_{i'} & k & 0^{\mathcal{K}} \\ \hline \vdots & \vdots & \vdots \end{array} \qquad\qquad \pi_B^k : \begin{array}{c||c|c} B & R & \neg R \\ \hline\hline b & k & 0^{\mathcal{K}} \end{array}$$

Clearly, Duplicator wins the game $G_1(\pi_A^k, \pi_B^k)$, as any possible strategy leads to winning of Duplicator. However, it holds that

$$\pi_A^k[\![\exists xRx]\!] = \sum_{i\in I} k \neq k = \pi_B^k[\![\exists xRx]\!] \text{ or}$$
$$\pi_A^k[\![\forall xRx]\!] = \prod_{i\in I} k \neq k = \pi_B^k[\![\forall xRx]\!],$$

which yields $\pi_A^k \not\equiv_1 \pi_B^k$ and completes the proof. $\square$

Notice that the theorem still holds if only model-defining $\mathcal{K}$-interpretations are considered. In case there is a semiring element $k \neq 0^{\mathcal{K}}$ witnessing the absence of full idempotence, the $\mathcal{K}$-interpretations π_A^k and π_B^k are model-defining and thus constitute an appropriate counterexample. Otherwise, the counterexample can be readily modified by substituting the valuations of the negated τ-literals over A and B by $1^{\mathcal{K}}$.

Concerning the question in which semirings m-equivalence is captured by the game G_m, we can exclude all semirings which are not fully idempotent, since the absence of full idempotence enables a restricted kind of counting, which is not taken into account in the standard Ehrenfeucht-Fraïssé game. We have shown that $\pi_A \equiv_m \pi_B$ is ensured if Duplicator wins $G_m(\pi_A, \pi_B)$ in any fully idempotent semiring $\mathcal{K}$, however this is not sufficient for m-equivalence of $\mathcal{K}$-interpretations to be captured by G_m. In addition, it must be possible to infer m-separability based on the winning of Spoiler. Making use of a lemma proven in [GM21], it can be shown that this is not possible for any fully idempotent semiring unless it is isomorphic to $\mathbb{B}$. Thus, $\mathbb{B}$ is, up to isomorphism, the only semiring such that G_m characterizes m-equivalence in semiring semantics.

Lemma 3.6. Let $\mathcal{K}$ be a fully idempotent semiring. For any $k_1, k_2 \in K$, it holds that $\pi_A^{k_1k_2} \equiv \pi_B^{k_1k_2}$ where $\pi_A^{k_1k_2}$ and $\pi_B^{k_1k_2}$ are defined as follows.

$\pi_A^{k_1k_2}$:

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_1	0	k_2	k_1	0
a_2	k_1	0	0	k_2
a_3	k_2	k_1	0	0
a_4	0	0	k_2	k_1

$\pi_B^{k_1k_2}$:

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	k_2	0	0	k_1
b_2	0	k_1	k_2	0
b_3	k_1	k_2	0	0
b_4	0	0	k_1	k_2

Theorem 3.7. Let $\mathcal{K}$ be a semiring (with infinitary operations Σ and Π). If $\pi_A \equiv_m \pi_B$ is equivalent to Duplicator winning $G_m(\pi_A, \pi_B)$ for all $\mathcal{K}$-interpretations π_A and π_B and all $m \in \mathbb{N}$, then $\mathcal{K}$ (with Σ and Π) is isomorphic to $\mathbb{B}$.

Proof. Let $\mathcal{K}$ (with Σ and Π) be a semiring such that $\pi_A \equiv_m \pi_B$ if, and only if, Duplicator wins $G_m(\pi_A, \pi_B)$ for all $\mathcal{K}$-interpretations π_A and π_B and all $m \in \mathbb{N}$. Due to theorem 3.5, $\mathcal{K}$ (with Σ and Π) must be fully idempotent. Suppose that $\mathcal{K}$ contains at least three elements and let $k_1, k_2 \in K$ be distinct and both non-zero. Spoiler wins the game $G_1(\pi_A^{k_1k_2}, \pi_B^{k_1k_2})$ where $\pi_A^{k_1k_2}$ and $\pi_B^{k_1k_2}$ are defined as in lemma 3.6. For instance, he can pick a_1, which cannot be duplicated in $\pi_B^{k_1k_2}$, since $k_1 \neq k_2$, $k_1 \neq 0^{\mathcal{K}}$ and $k_2 \neq 0^{\mathcal{K}}$. But lemma 3.6 yields that $\pi_B^{k_1k_2} \equiv \pi_B^{k_1k_2}$, so, in particular, $\pi_B^{k_1k_2} \equiv_1 \pi_B^{k_1k_2}$. This is a contradiction to the assumption that $\pi_A \equiv_m \pi_B$ if, and only if, Duplicator wins $G_m(\pi_A, \pi_B)$ for all $\mathcal{K}$-interpretations

π_A and π_B and all $m \in \mathbb{N}$. Hence, $\mathcal{K}$ can only contain two elements, namely the neutral elements $0^{\mathcal{K}}$ and $1^{\mathcal{K}}$. The semiring axioms determine the behavior of addition and multiplication in $\mathcal{K}$ almost completely as follows.

$+^{\mathcal{K}}$	$0^{\mathcal{K}}$	$1^{\mathcal{K}}$
$0^{\mathcal{K}}$	$0^{\mathcal{K}}$	$1^{\mathcal{K}}$
$1^{\mathcal{K}}$	$1^{\mathcal{K}}$	$\star$

$\cdot^{\mathcal{K}}$	$0^{\mathcal{K}}$	$1^{\mathcal{K}}$
$0^{\mathcal{K}}$	$0^{\mathcal{K}}$	$0^{\mathcal{K}}$
$1^{\mathcal{K}}$	$0^{\mathcal{K}}$	$1^{\mathcal{K}}$

Hence, only $1^{\mathcal{K}} +^{\mathcal{K}} 1^{\mathcal{K}}$ might differ from addition in $\mathbb{B}$. But as $\mathcal{K}$ is known to be fully idempotent, it must hold that $1^{\mathcal{K}} +^{\mathcal{K}} 1^{\mathcal{K}} = 1^{\mathcal{K}}$, thus $\mathcal{K} \cong \mathbb{B}$. If $\mathcal{K}$ has infinitary operations, they must also be idempotent. Thus, because of invariance under partitions and the fact that finite addition and multiplication respect the isomorphism $0^{\mathcal{K}} \mapsto 0^{\mathbb{B}}$, $1^{\mathcal{K}} \mapsto 1^{\mathbb{B}}$, it follows immediately that the infinitary operations respect the isomorphism as well. $\square$

It is noticeable that the counterexamples used to show that $\mathcal{K}$ is fully idempotent and only contains two elements share certain similarities. For instance, they only rely on unary relations and in both cases, the game G_1 does not provide the desired result. Moreover, one can always find model-defining counterexamples. Therefore, the premise in theorem 3.7 can be refined and we can already conclude that $\mathcal{K}$ must be isomorphic to $\mathbb{B}$ if the game G_1 characterizes 1-equivalence for all model-defining $\mathcal{K}$-interpretations over a vocabulary consisting of unary relation symbols.

Whenever we can infer m-equivalence from a winning strategy for Duplicator in G_m for all $m \in \mathbb{N}$ in a certain semiring $\mathcal{K}$, concluding elementary equivalence based on the game G is possible as well. This is because Duplicator wins G if, and only if, she wins G_m for all $m \in \mathbb{N}$ and that elementary equivalence coincides with m-equivalence for all $m \in \mathbb{N}$, just like for classical semantics. Combining this observation with theorem 3.5 yields the following corollary.

Corollary 3.8. If $\mathcal{K}$ is fully idempotent and Duplicator wins $G(\pi_A, \pi_B)$, then $\pi_A \equiv \pi_B$ must hold.

Whether full idempotence is not just a sufficient but also a necessary condition, as observed for the game G_m, is still open. Even though m moves are generally not sufficient for Spoiler to win based on a separating formula of quantifier rank m for semirings that are not fully idempotent, permitting a fixed number of moves that might be greater that m may ensure the winning of Spoiler. For example, on the introductory $\mathbb{N}$-interpretations π_A^1 and π_B^1, Spoiler is not able to win G_1, but adding one additional move makes Spoiler win. Further, it is not clear whether

Duplicator winning G implies elementary equivalence for all semirings with appropriate infinitary operations. We postpone these questions for now and discuss them within the following subsections, which are dedicated to the transferability of the games G and G_m for certain common semirings.

3.1.1 Min-Max Semirings

The classical logical connectives $\vee$ and $\wedge$ can be considered as the binary operations maximum and minimum applied to elements in $\{0, 1\}$ such that $0 < 1$. This observation suggests to generalize the Boolean semiring by using maximum as addition and minimum as multiplication, both operating on an arbitrary linearly ordered set K with ending points.

Definition 3.9. A linearly ordered set $(K, \leq)$ with least element k_0 and greatest element k_1 induces a *min-max semiring* $\mathcal{K} = (K, \max, \min, k_0, k_1)$ where max and min refer to the underlying order.

Practically relevant min-max semiring are, for instance, $\mathbb{F} = ([0, 1]_{\mathbb{R}}, \max, \min, 0, 1)$ we refer to as the *fuzzy semiring*. The access control semiring introduced earlier, given by $\mathbb{A} = (\{P, C, S, T, 0\}, \min, \max, 0, P)$, is in fact a min-max semiring as well, as it is induced by the inverted order on the access levels. Min-max semirings can be considered a generalization of the Boolean semiring which is still similar to $\mathbb{B}$. Unlike other semirings, min-max semirings share algebraic properties such as full idempotence and absorption with $\mathbb{B}$ and therefore maintain logical equivalences such as $\varphi \wedge \varphi \equiv \varphi$. Further, each min-max semiring is naturally ordered by the order it is induced by. Hence, the natural order is in fact a linear order, so any two elements are comparable. To evaluate formulae with elements from a min-max semiring on infinite interpretations as well, we additionally assume that any subset $K' \subseteq K$ has an infimum and supremum in $\mathcal{K}$ and define the infinitary operations as the supremum of the finite subsums and the infimum of the finite subproducts, which correspond to supremum and infimum with respect to the underlying order. So with any family $(k_i)_{i \in I}$ of elements from a min-max semiring $\mathcal{K}$, we associate

$$\sum_{i \in I} k_i := \sup_{\substack{I' \subseteq I \\ \text{finite}}} \overset{\text{fin}}{\sum_{i \in I'}} k_i = \sup_{i \in I} k_i \quad \text{and} \quad \prod_{i \in I} k_i := \inf_{\substack{I' \subseteq I \\ \text{finite}}} \overset{\text{fin}}{\prod_{i \in I'}} k_i = \inf_{i \in I} k_i.$$

This definition ensures that on each min-max semiring, the infinitary operations are fully idempotent as well. Hence, we can apply theorem 3.5 and the corresponding corollary 3.8 to any min-max semiring $\mathcal{K}$ and infer that any two $\mathcal{K}$-interpretations are m-equivalent if Duplicator wins the game G_m, and that they are elementarily equivalent in case Duplicator wins G.

However, the standard game cannot be used as a tool to prove the existence of a separating sentence of quantifier rank at most m, so to derive m-separability of $\mathcal{K}$-interpretations where $\mathcal{K}$ is a min-max semiring. Unlike Boolean quantifiers or, correspondingly, the moves of Spoiler, quantifiers in semiring semantics cannot pick out an arbitrary element from the universe of one of the $\mathcal{K}$-interpretations, as indicated by the introductory interpretations π_A^2 and π_B^2.

Proposition 3.10. There is a min-max semiring $\mathcal{K}$ and $\mathcal{K}$-interpretations π_A, π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$, while $\pi_A \equiv_1 \pi_B$.

Proof. Let $\mathcal{K}_4 := (\{0, 1, 2, 3, 4\}, \max, \min, 0, 4)$ be induced by the usual order on $\{0, 1, 2, 3, 4\}$ and recall the $\mathcal{K}_4$-interpretations π_A^2 and π_B^2.

π_A^2 :

A	R	$\neg R$
a_1	1	0
a_2	2	0
a_3	4	0

π_B^2 :

B	R	$\neg R$
b_1	1	0
b_2	3	0
b_3	4	0

Spoiler wins the game $G_1(\pi_A^2, \pi_B^2)$ by picking $a_2 \in A$. The valuation 2 with regard to R cannot be duplicated in π_B^1, which is why Spoiler wins any play according to this strategy.

In order to show that π_A^2 and π_B^2 are 1-equivalent, consider for $0 \leq k < 4$ the mapping $h_k \colon \mathcal{K}_4 \to \mathbb{B}$ with $h_k \colon x \mapsto 0$ if, and only if, $x \leq k$. Each of these functions is a homomorphism from $\mathcal{K}_4$ to $\mathbb{B}$, because for $0 \leq k < 4$ it holds that

(1) $h_k(0) = 0$ and $h_k(4) = 1$,

(2) $h_k(x +^{\mathcal{K}_4} y) = h_k(\max(x, y)) = \left\{ \begin{array}{ll} 0, & x \leq k \wedge y \leq k \\ 1, & x > k \vee y > k \end{array} \right\} = h_k(x) +^{\mathbb{B}} h_k(y)$ and

(3) $h_k(x \cdot^{\mathcal{K}_4} y) \;=\; h_k(\min(x, y)) = \left\{ \begin{array}{ll} 0, & x \leq k \vee y \leq k \\ 1, & x > k \wedge y > k \end{array} \right\} = h_k(x) \cdot^{\mathbb{B}} h_k(y)$,

where $x, y \in K$. The set $H := \{h_k \colon 0 \leq k < 4\}$ is a separating set of homomorphisms, because for any $x, y \in K$ with $x \neq y$ it holds that

$$h_{\min(x,y)}(x) = 0 \neq 1 = h_{\min(x,y)}(y).$$

For $k \in \{0, 1, 3\}$ we have that $(h_k \circ \pi_A^2) \cong (h_k \circ \pi_B^2)$, which yields $(h_k \circ \pi_A^2) \equiv (h_k \circ \pi_B^2)$ by the isomorphism lemma. This is not true for $k = 2$, but Duplicator wins the game $G_1(h_2 \circ \pi_A^2, h_2 \circ \pi_B^2)$ by replying b_i to each a_i and vice versa, except for a_2, which is answered with b_1, and b_2, which she counters with a_3.

$h_2 \circ \pi^2_A$:

A	R	$\neg R$
a_1	0	0
a_2	0	0
a_3	1	0

$h_2 \circ \pi^2_B$:

B	R	$\neg R$
b_1	0	0
b_2	1	0
b_3	1	0

As $\mathbb{B}$ is fully idempotent, we can apply theorem 3.5 and infer $(h_2 \circ \pi^2_A) \equiv_1 (h_2 \circ \pi^2_B)$. Hence, in particular, it holds that $(h_k \circ \pi^2_A) \equiv_1 (h_k \circ \pi^2_B)$ for $0 \leq k < 4$. Following lemma 2.21, this implies $\pi^2_A \equiv_1 \pi^2_B$. $\square$

The counterexample demonstrates that in semiring semantics, quantifiers behave in a fundamentally different way than in Boolean semantics, even though min-max semirings are still quite similar to the Boolean semiring. Whereas in the Boolean case quantifiers allow access to any element, in min-max semirings a quantifier can only filter out those elements that have a maximum or minimum valuation with regard to some property. Any valuation in between minimum and maximum remains concealed and does not influence the valuation of any sentence with quantifier rank 1. In this example, increasing the quantifier rank causes the $\mathcal{K}_4$-interpretations π^2_A and π^2_B to become separable. The formula

$$\varphi := \exists x \exists y (x \neq y \wedge Rx \wedge Ry)$$

allows to filter out not the minimum or maximum valuation with respect to R but the second greatest valuation. This observation suggests that the problems arising when applying the game to min-max semirings is due to the number of moves, which does not suit the quantifier rank. But it turns out that the problem is more fundamental, as there are even elementarily equivalent $\mathcal{K}$-interpretations for a min-max semiring $\mathcal{K}$ where every strategy of Spoiler in G is a winning strategy. From the counterexample introduced in [GM21] to prove that elementary equivalence does not imply isomorphism in min-max semirings, we derive that the game G cannot be used to prove separability in min-max semirings.

Proposition 3.11. There is a min-max semiring $\mathcal{K}$ and $\mathcal{K}$-interpretations π_A, π_B such that Spoiler wins $G(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$.

Proof. We construct $\mathcal{K}_3$-interpretations π^3_A and π^3_B based on the min-max semiring $\mathcal{K}_3 := (\{0, 1, 2, 3, 4\}, \max, \min, 0, 3)$ which relies on the usual order of the elements.

π^3_A :

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_1	1	3	0	0
a_2	2	1	0	0
a_3	3	2	0	0

π^3_B :

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	3	1	0	0
b_2	1	2	0	0
b_3	2	3	0	0

Spoiler wins the game $G(\pi_A^3, \pi_B^3)$, for instance, he can choose $m = 1$ and $a_1 \in A$, as there is no element in $b \in B$ such that $\pi_B^3(R_1 b) = 1$ and $\pi_B^3(R_2 b) = 3$.

Analogous to the previous proof, it can be shown that π_A^3 and π_B^3 are elementarily equivalent by making use of the homomorphisms $h_k \colon \mathcal{K}_3 \to \mathbb{B}$ with $h_k \colon x \mapsto 0$ if, and only if, $x \leq k$ for $0 \leq k < 3$. For each such k it holds that $(h_k \circ \pi_A^3) \cong (h_k \circ \pi_B^3)$. Applying the isomorphism lemma yields that $(h_k \circ \pi_A^3) \equiv (h_k \circ \pi_B^3)$ for $0 \leq k < 3$. Hence, $\pi_A^3 \equiv \pi_B^3$ must hold. □

In the game $G(\pi_A^3, \pi_B^3)$, Duplicator is able to duplicate any valuation with regard to R_1 or R_2, separately, but when considering R_1 and R_2 simultaneously, none of the valuations can be duplicated in the respective other structure. However, this does not influence the valuation of any sentence. This observation suggests that Spoiler has to reveal a literal he will challenge before Duplicator chooses her answer in order to capture the elementary equivalence of π_A^3 and π_B^3. But this would cause Duplicator to win more often than desired, as a slight modification of π_A^3 and π_B^3 illustrates.

π_A^4 :

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_1	1	3	0	0
a_2	2	1	0	0
a_3	3	2	0	0

π_B^4 :

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	3	2	0	0
b_2	1	1	0	0
b_3	2	3	0	0

The modified $\mathcal{K}_3$-interpretations π_A^4 and π_B^4 can be separated by the sentence $\forall x(R_1 x \vee R_2 x)$, although each valuation with regard to R_1 and R_2 is still individually duplicable in the respective other interpretation. Hence, Spoiler would rather have to be able to commit to a positive Boolean combination of literals before Duplicator chooses her answer. While this observation provides first insights that it is not straightforward to adjust the game rules, we will discuss possible modifications in more detail in chapter 5 and now return to the applicability of the standard game to further semirings.

3.1.2 The Viterbi Semiring $\mathbb{V}$

Besides min-max semirings, the Viterbi semiring $\mathbb{V} = ([0,1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ constitutes another absorptive application semiring, which is thus also idempotent. By contrast, multiplicative idempotence is not fulfilled. The Viterbi semiring is isomorphic to the tropical semiring $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$, which will thus be examined implicitly, as all results obtained for the Viterbi semiring can be directly transferred via the isomorphism $x \mapsto -\ln(x)$. $\mathbb{V}$ is naturally ordered by the usual order on $[0,1]_{\mathbb{R}}$, which is not just a partial order but a dense linear order. These properties enable an axiomatization of any finite $\mathbb{V}$-interpretation up to isomorphism. Hence, the Viterbi semiring witnesses that there are indeed semirings which are more complex than the Boolean semiring but still ensure the separability of any two finite interpretations which differ not just by isomorphism [GM21]. So, the question arises whether this result can be extended and a winning strategy of Spoiler in G_m also ensures m-separability of finite $\mathbb{V}$-interpretations, or whether it can be transferred to infinite $\mathbb{V}$-interpretations. In order to permit infinite universes as well, it remains to specify the definition of the infinitary operations. As before, we consider the supremum of the finite subsums as infinitary summation and the infimum of the finite subproducts as infinitary multiplication. Multiple of the following observations rely on invariance under partitions. Thus, we first verify that this property is satisfied by the infinitary operations in $\mathbb{V}$ or, more generally, in any semiring with analogous infinitary operations.

Lemma 3.12. Let $\mathcal{K}$ be a semiring with infinitary operations given by

$$\sum_{i\in I} k_i := \sup_{\substack{I'\subseteq I\\ \text{finite}}} \sum_{i\in I'} k_i \quad \text{and} \quad \prod_{i\in I} k_i := \inf_{\substack{I'\subseteq I\\ \text{finite}}} \prod_{i\in I'} k_i$$

for all families $(k_i)_{i\in I}$ in $\mathcal{K}$. Then, for each $(k_i)_{i\in I}$ and any partition $\mathcal{P}$ of I, it holds that

$$\sum_{i\in I} k_i = \sum_{S\in\mathcal{P}} \sum_{i\in S} k_i \text{ and } \prod_{i\in I} k_i = \prod_{S\in\mathcal{P}} \prod_{i\in S} k_i.$$

Proof. We only prove the claim for infinitary multiplication in $\mathcal{K}$, as the reasoning for infinitary summation is analogous. We first show that the claim is true for any partition of cardinality 2 and generalize this to arbitrary cardinalities by induction afterwards. Let $\mathcal{P} = \{S_1, S_2\}$ be a partition of I and $k_\infty := \prod_{i\in I} k_i$. As finite multiplication in $\mathcal{K}$ is invariant under partitions due to associativity, we have that

$$\prod_{i\in S_1'} k_i \cdot \prod_{i\in S_2'} k_i = \prod_{i\in S_1'\dot\cup S_2'} k_i \geq k_\infty$$

for any finite $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$. Hence, k_∞ is a lower bound on

$$\prod_{i\in S_1} k_i \cdot \prod_{i\in S_2} k_i = \inf_{\substack{S_1'\subseteq S_1\\ \text{finite}}} \prod_{i\in S_1'} k_i \cdot \inf_{\substack{S_2'\subseteq S_2\\ \text{finite}}} \prod_{i\in S_2'} k_i.$$

If there was some $\varepsilon > 0$ such that $\prod_{i\in S_1'} k_i \cdot \prod_{i\in S_2'} k_i \geq k_\infty + \varepsilon$ for all finite $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$, then for any finite $I' \subseteq I$ we would have that

$$\prod_{i\in I'} k_i = \prod_{i\in S_1\cap I'} k_i \cdot \prod_{i\in S_2\cap I'} k_i \geq k_\infty + \varepsilon,$$

yielding a contradiction. Hence, it must hold that $k_\infty = \prod_{i\in S_1} k_i \cdot \prod_{i\in S_2} k_i$.

By induction, it readily follows that infinitary multiplication in $\mathcal{K}$ is invariant under finite partitions $\mathcal{P} = S_1\dot\cup\ldots\dot\cup S_n$. Hence, it remains to show the claim for infinite partitions $\mathcal{P} = \{S_j : j \in J\}$ of I where J is some additional infinite index set. For any finite $J' \subseteq J$, we write $S_{J'} = \bigcup_{j\in J'} S_j$. Using invariance under finite partitions ($\star$), we can conclude that

$$\prod_{j\in J}\prod_{i\in S_j} k_i = \inf_{\substack{J'\subseteq J\\ \text{finite}}} \prod_{j\in J'}\prod_{i\in S_j} k_i \overset{(\star)}{=} \inf_{\substack{J'\subseteq J\\ \text{finite}}} \prod_{i\in S_{J'}} k_i = \inf_{\substack{J'\subseteq J\\ \text{finite}}} \inf_{\substack{S'\subseteq S_{J'}\\ \text{finite}}} \prod_{i\in S'} k_i = \inf_{\substack{I'\subseteq I\\ \text{finite}}} \prod_{i\in I'} k_i = \prod_{i\in I} k_i.$$

Thus, infinitary multiplication in $\mathcal{K}$ is also invariant under infinite partitions. □

With invariance under partitions being verified, we are ready to examine the transferability of the games G_m and G to $\mathbb{V}$-interpretations.

Completeness of G_m. Even though the absence of isomorphism implies separability for finite $\mathbb{V}$-interpretations, the moves of Spoiler do not correspond to separability by a formula of certain quantifier rank. The game G_m as a method to show m-separability already fails at simple counterexamples relying on a single unary relation R. Under the assumption that any negated τ-literal is valuated with 0, it can be proven that it suffices if sum and product of the valuations of the unnegated τ-literals coincide in order to obtain 1-equivalence, independent of the exact valuations.

Proposition 3.13. For any $\mathbb{V}$-interpretations π_A, π_B over vocabulary $\tau = \{R\}$ consisting of a unary relation symbol such that

(1) $\pi_A(\neg Ra) = \pi_B(\neg Rb) = 0$ for all $a \in A$ and $b \in B$,

(2) $\sup_{a\in A} \pi_A(Ra) = \sup_{b\in B} \pi_B(Rb)$ and

(3) $\prod_{a\in A} \pi_A(Ra) = \prod_{b\in B} \pi_B(Rb)$,

it must hold that $\pi_A \equiv_1 \pi_B$.

Proof. Let $\varphi(x)^n$ denote the formula $\underbrace{\varphi(x)\wedge\cdots\wedge\varphi(x)}_{n\text{ times}}$ and let

$$\Phi := \{x = x, x \neq x\} \cup \{Rx^n : n \in \mathbb{N}_{>0}\}.$$

We show by induction on the structure of the formula that for all quantifier-free $\psi(x)$, there is some $\psi^*(x) \in \Phi$ such that

$$\pi_A[\![\psi(a)]\!] = \pi_A[\![\psi^*(a)]\!] \text{ for all } a \in A \text{ and } \pi_B[\![\psi(b)]\!] = \pi_B[\![\psi^*(b)]\!] \text{ for all } b \in B.$$

Case 1. Each literal $\varphi(x)$ is already included in Φ, except for $\neg Rx$. For the remaining case $\neg Rx$ we have that

$$\pi_A(\neg Ra) = 0 = \pi_A(a \neq a) \text{ and}$$
$$\pi_B(\neg Rb) = 0 = \pi_B(b \neq b)$$

for all $a \in A$ and $b \in B$, which completes the base case.

Case 2. Suppose that $\psi(x) = \psi_1(x) \circ \psi_2(x)$ with $\circ \in \{\wedge, \vee\}$. By induction hypothesis, there are $\psi_1^*(x), \psi_2^*(x) \in \Phi$ fulfilling the required property with regard to $\psi_1(x)$ and $\psi_2(x)$. Depending on $\psi_1^*(x)$ and $\psi_2^*(x)$, the formulae $(\psi_1 \vee \psi_2)^*(x)$ and $(\psi_1 \wedge \psi_2)^*(x)$ can be chosen according to the following table.

$\psi_1^*(x)$	$\psi_2^*(x)$	$(\psi_1 \vee \psi_2)^*(x)$	$(\psi_1 \wedge \psi_2)^*(x)$
$x = x$	$\psi_2^*(x)$	$x = x$	$\psi_2^*(x)$
$x \neq x$	$\psi_2^*(x)$	$\psi_2^*(x)$	$x \neq x$
Rx^n	Rx^m	$Rx^{\min(n,m)}$	Rx^{n+m}

The entries $\psi_2^*(x)$ in the first two lines are supposed to reflect that any formula in Φ can be inserted, so, for instance, $(x = x \wedge \psi_2(x))^* = \psi_2^*(x)$ for any quantifier-free formula $\psi_2(x)$. For all remaining cases which are not included in the table, we simply set $(\psi_1 \circ \psi_2)^*(x) = (\psi_2 \circ \psi_1)^*(x)$ where $\circ \in \{\vee, \wedge\}$. The choice of $\psi^*(x)$ in case $\psi_1^*(x)$ or $\psi_2^*(x)$ is an (in)equality relies on the semiring equalities $1 \cdot k = \max(0, k) = k$, $\max(1, k) = 1$ and $0 \cdot k = 0$, which hold for all $k \in [0, 1]_{\mathbb{R}}$. In addition, we have that $k^n \cdot k^m = k^{n+m}$ and $\max(k^n, k^m) = k^{\min(n,m)}$ for $k \in [0, 1]_{\mathbb{R}}$ justifying the choice of $\psi^*(x)$ if $\psi_1^*(x) = Rx^n$ and $\psi_2^*(x) = Rx^m$.

If there is a sentence of quantifier rank 1 which separates π_A and π_B, then there must be a separating sentence of the form $Qx\psi(x)$ with $Q \in \{\exists, \forall\}$ due to lemma 2.19 and the fact that every literal contains at least one variable, which must be quantified in a sentence. But this implies that π_A and π_B can be separated by some formula $Qx\psi^*(x)$ where $\psi^*(x) \in \Phi$. This yields a contradiction, since

$$\pi_A[\![\exists x(x = x)]\!] = \pi_A[\![\forall x(x = x)]\!] = 1 = \pi_B[\![\forall x(x = x)]\!] = \pi_B[\![\exists x(x = x)]\!] \text{ and}$$
$$\pi_A[\![\exists x(x \neq x)]\!] = \pi_A[\![\forall x(x \neq x)]\!] = 0 = \pi_B[\![\forall x(x \neq x)]\!] = \pi_B[\![\exists x(x \neq x)]\!].$$

Further, it holds that $k \leq \ell$ if, and only if, $k^n \leq \ell^n$ for all $k, \ell \in [0, 1]_{\mathbb{R}}$ and all $n \in \mathbb{N}_{>0}$. Therefore, we can infer for each such n that

$$\sup_{a \in A} \pi_A(Ra)^n = \big(\sup_{a \in A} \pi_A(Ra)\big)^n = \big(\sup_{b \in B} \pi_B(Rb)\big)^n = \sup_{b \in B} \pi_B(Rb)^n.$$

Analogously, it must hold due to invariance under partitions of infinitary multiplication in $\mathbb{V}$ that

$$\prod_{a\in A} \pi_A(Ra)^n = \Big(\prod_{a\in A} \pi_A(Ra)\Big)^n = \Big(\prod_{b\in B} \pi_B(Rb)\Big)^n = \prod_{b\in B} \pi_B(Rb)^n.$$

Hence, we can conclude that

$$\begin{aligned} \pi_A[\![\exists x(Rx^n)]\!] &= \sup_{a\in A} \pi_A(Ra)^n = \sup_{b\in B} \pi_B(Ra)^n = \pi_B[\![\exists x(Rx^n)]\!] \text{ and} \\ \pi_A[\![\forall x(Rx^n)]\!] &= \prod_{a\in A} \pi_A(Ra)^n = \prod_{b\in B} \pi_B(Ra)^n = \pi_B[\![\forall x(Rx^n)]\!], \end{aligned}$$

so a separating formula of quantifier rank 1 cannot exist, i.e., $\pi_A \equiv_1 \pi_B$. □

Making use of this proposition, it is easy to construct a counterexample illustrating that m-separability cannot be inferred from G_m.

π_A^5 :

A	R	$\neg R$
a_1	1	0
a_2	0.5	0
a_3	0.5	0

π_B^5 :

B	R	$\neg R$
b_1	1	0
b_2	0.25	0

Since for all $a \in A$ and $b \in B$ it holds that

$$\begin{aligned} \pi_A^5(\neg Ra) &= 0 = \pi_B^5(\neg Rb), \\ \sup_{a\in A} \pi_A^5(Ra) &= 1 = \sup_{b\in B} \pi_B^5(Rb) \text{ and} \\ \prod_{a\in A} \pi_A^5(Ra) &= 0.25 = \prod_{b\in B} \pi_B^5(Rb), \end{aligned}$$

we can conclude $\pi_A^5 \equiv_1 \pi_B^5$. However, Spoiler clearly wins $G_1(\pi_A^5, \pi_B^5)$, e.g., by choosing $a_2 \in A$, which cannot be duplicated in π_B^5.

Proposition 3.14. There are finite $\mathbb{V}$-interpretations π_A and π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv_1 \pi_B$.

Note that if the vocabulary consists of multiple unary relation symbols, it does not suffice to demand the requirements in proposition 3.13 for each of the relations separately in order to deduce 1-equivalence. This is because sentences of quantifier rank 1 also allow multiplication over the maximum valuation of certain τ-literals over the same instantiation or filtering out the maximum product. As an example, consider the following $\mathbb{V}$-interpretations.

π_A^6 :

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_1	1	0.5	0	0
a_2	0.5	1	0	0
a_3	0.5	0.5	0	0

π_B^6 :

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	1	1	1	0
b_2	0.25	0.25	0	0

For $i \in \{1,2\}$ we still have that

$$\sup_{a\in A} \pi_A^6(R_i a) = 1 = \sup_{b\in B} \pi_B^6(R_i b) \quad \text{and} \quad \prod_{a\in A} \pi_A^6(R_i a) = 0.25 = \prod_{b\in B} \pi_B^6(R_i b),$$

while all negated τ-literals over A and B are valuated with 0. However, the $\mathbb{V}$-interpretations are not 1-equivalent, e.g.,

$$\pi_A^6[\![\forall x(R_1x \vee R_2x)]\!] = 0.5 \neq 0.25 = \pi_B^6[\![\forall x(R_1x \vee R_2x)]\!] \text{ and}$$
$$\pi_A^6[\![\exists x(R_1x \wedge R_2x)]\!] = 0.5 \neq 1 = \pi_B^6[\![\exists x(R_1x \wedge R_2x)]\!].$$

Hence, proposition 3.13 does not provide a general characterization of 1-equivalence for $\mathbb{V}$-interpretations. Nevertheless, it establishes a sufficient condition for 1-equivalence for $\mathbb{V}$-interpretations over vocabularies consisting of a single unary relation symbol, which illustrates that 1-separability cannot be inferred from a winning strategy for Spoiler in G_1.

Completeness of G. Even though the result stating that non-isomorphic finite $\mathbb{V}$-interpretations can always be separated could not be generalized regarding m-separability and the game G_m, it has immediate consequences for the game G if only finite $\mathbb{V}$-interpretations are considered. If Spoiler wins the game $G(\pi_A, \pi_B)$, then π_A and π_B cannot be isomorphic, as Spoiler can pick all elements in the universe of one of the structures. Thus, the finite $\mathbb{V}$-interpretations actually have to be separable by some sentence.

Proposition 3.15. Given finite $\mathbb{V}$-interpretations π_A and π_B, it must hold that $\pi_A \not\equiv \pi_B$ if Spoiler has a winning strategy for $G(\pi_A, \pi_B)$.

As an example, the $\mathbb{V}$-interpretations π_A^5 and π_B^5, which are 1-equivalent, although Spoiler wins G_1, must be separable by a sentence with larger quantifier rank. Similar to the $\mathcal{K}_4$-interpretations in the previous section, π_A^5 and π_B^5 can be separated by $\varphi := \exists x \exists y(x \neq y \wedge Rx \wedge Ry)$, which is evaluated by the product of the two largest distinct valuations with respect to R. In this manner, we obtain that

$$\pi_A^5[\![\varphi]\!] = 1 \cdot 0.5 = 0.5 \neq 0.25 = \pi_B^5[\![\varphi]\!].$$

Whether proposition 3.15 can be generalized to infinite $\mathbb{V}$-interpretations, is still open. Therefore, we examine next the question of whether separability can be inferred from the winning of Spoiler in G for $\mathbb{V}$-interpretations of arbitrary cardinality, and answer it in the negative. In particular, we show that a single unary relation R suffices to create a counterexample, that is, $\mathbb{V}$-interpretations we refer to as π_A^7 and π_B^7. In the same manner as we constructed a sentence φ separating the $\mathbb{V}$-interpretations π_A^5 and π_B^5, it is possible to construct sentences of quantifier rank n which are evaluated by the product of the n largest valuations with regard to R, as shown in [Mrk20]. Hence, to obtain elementarily equivalent π_A^7 and π_B^7, infinitely many elements are contained in both A and B, each of which is valuated with 1 with respect to R. In order to ensure the winning of Spoiler, we add further elements to both A and B resulting in distinct valuations in π_A^7 and π_B^7 which are smaller than 1. By incorporating the same valuation infinitely often, we make sure that product over all valuations with regard to R will be 0 in both π_A^7 and π_B^7.

π_A^7 :

A	R	$\neg R$
a_1	0.1	0
a_2	1	0
a_3	0.1	0
a_4	1	0
$\vdots$	$\vdots$	$\vdots$

π_B^7 :

B	R	$\neg R$
b_1	0.01	0
b_2	1	0
b_3	0.01	0
b_4	1	0
$\vdots$	$\vdots$	$\vdots$

Clearly, Spoiler wins the game $G(\pi_A^7, \pi_B^7)$. In fact, he already wins $G_1(\pi_A^7, \pi_B^7)$, for instance, by choosing $a_1 \in A$. To prove that π_A^7 and π_B^7 constitute an appropriate counterexample, it thus suffices to show the elementary equivalence of the $\mathbb{V}$-interpretations. For this purpose, we will make use of a generalized version of the isomorphism lemma which does not relate to the equality of valuations but to the natural order.

Definition 3.16. Given a naturally ordered semiring $\mathcal{K}$, let π_A and π_B be $\mathcal{K}$-interpretations with elements $\bar{a} = (a_1, \dots, a_n) \in A^n$ and $\bar{b} = (b_1, \dots, b_n) \in B^n$. We say that $(\pi_A, \bar{a})$ is *$\leq$-isomorphic* to $(\pi_B, \bar{b})$ if there is a bijection $\sigma\colon A \to B$ with

(1) $\sigma(a_i) = b_i$ for all $1 \leq i \leq n$ and

(2) $\pi_A(L(\bar{a})) \leq \pi_B(L(\bar{b}))$ for all $L(\bar{x}) \in \mathrm{Lit}_n(\tau)$.

If both conditions are satisfied, we refer to the mapping σ as an *$\leq$-isomorphism* from $(\pi_A, \bar{a})$ to $(\pi_B, \bar{b})$.

Lemma 3.17. Let $\mathcal{K}$ be naturally ordered, π_A and π_B be $\mathcal{K}$-interpretations and $\bar{a} = (a_1, \dots, a_n) \in A^n$, $\bar{b} = (b_1, \dots, b_n) \in B^n$. If π_A or π_B is infinite, further assume that the infinitary operations are monotone. If $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$ are $\leq$-isomorphic, then it holds that $\pi_A[\![\varphi(\bar{a})]\!] \leq \pi_B[\![\varphi(\bar{b})]\!]$ for all formulae $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$.

Proof. We prove the claim for all $\bar{a} = (a_1, \ldots, a_n) \in A^n$, $\bar{b} = (b_1, \ldots, b_n) \in B^n$ and $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ by induction on the structure of $\varphi(\bar{x})$.

Case 1. Since $(\pi_A, \bar{a})$ is $\leq$-isomorphic to $(\pi_B, \bar{b})$ by assumption, there must be a bijection $\sigma\colon A \to B$ with $\sigma(a_i) = b_i$ for all $1 \leq i \leq n$. This immediately implies $a_i = a_j$ if, and only if, $b_i = b_j$, thus $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ for all $\varphi(\bar{x}) = x_i \circ x_j$ with $\circ \in \{=, \neq\}$ and $1 \leq i, j \leq n$.

Case 2. If $\varphi(\bar{x}) \in \{Rx_{i_1} \ldots x_{i_r}, \neg Rx_{i_1} \ldots x_{i_r}\}$ with $R \in \tau$, $\mathrm{arity}(R) = r$ and $1 \leq x_{i_1}, \ldots, x_{i_r} \leq n$, it immediately follows from the existence of a $\leq$-isomorphism from $(\pi_A, \bar{a})$ to $(\pi_B, \bar{b})$ that $\pi_A[\![\varphi(\bar{a})]\!] \leq \pi_B[\![\varphi(\bar{b})]\!]$.

Case 3. For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$ where $\circ \in \{\vee, \wedge\}$, applying the induction hypothesis yields that $\pi_A[\![\psi(\bar{a})]\!] \leq \pi_B[\![\psi(\bar{b})]\!]$ and $\pi_A[\![\vartheta(\bar{a})]\!] \leq \pi_B[\![\vartheta(\bar{b})]\!]$. Following lemma 2.5, addition and multiplication are monotone in $\mathcal{K}$, since $\mathcal{K}$ is naturally ordered by assumption. Thus,

$$\begin{aligned}
\pi_A[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] + \pi_A[\![\vartheta(\bar{a})]\!] \\
&\leq \pi_B[\![\psi(\bar{b})]\!] + \pi_B[\![\vartheta(\bar{b})]\!] = \pi_B[\![\psi(\bar{b}) \vee \vartheta(\bar{b})]\!] \text{ and} \\
\pi_A[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!] &= \pi_A[\![\psi(\bar{a})]\!] \cdot \pi_A[\![\vartheta(\bar{a})]\!] \\
&\leq \pi_B[\![\psi(\bar{b})]\!] \cdot \pi_B[\![\vartheta(\bar{b})]\!] \quad = \pi_B[\![\psi(\bar{b}) \wedge \vartheta(\bar{b})]\!]
\end{aligned}$$

must hold as well.

Case 4. For $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$, let σ be a $\leq$-isomorphism from $(\pi_A, \bar{a})$ to $(\pi_B, \bar{b})$. By induction hypothesis, $\pi_A[\![\psi(\bar{a}, a)]\!] \leq \pi_B[\![\psi(\bar{b}, \sigma(a))]\!]$ for all $a \in A$. As the finite operations are monotone on any naturally ordered semiring and the infinitary operations are monotone by assumption, it can be inferred that

$$\begin{aligned}
\pi_A[\![\exists x\psi(\bar{a}, x)]\!] &= \sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \\
&\leq \sum_{a \in A} \pi_B[\![\psi(\bar{b}, \sigma(a))]\!] \\
&= \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \qquad = \pi_B[\![\exists x\psi(\bar{b}, x)]\!] \text{ and} \\
\pi_A[\![\forall x\psi(\bar{a}, x)]\!] &= \prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \\
&\leq \prod_{a \in A} \pi_B[\![\psi(\bar{b}, \sigma(a))]\!] \\
&= \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \qquad = \pi_B[\![\forall x\psi(\bar{b}, x)]\!].
\end{aligned}$$

$\square$

Hence, it suffices to prove that the infinitary operations in $\mathbb{V}$ are monotone in order to apply the modified isomorphism lemma to the infinite $\mathbb{V}$-interpretations π_A^7 and π_B^7.

Lemma 3.18. Let $\mathcal{K}$ be a naturally ordered semiring with infinitary operations defined as

$$\sum_{i\in I} k_i := \sup_{\substack{I'\subseteq I\\ \text{finite}}} \sum_{i\in I'} k_i \quad \text{and} \quad \prod_{i\in I} k_i := \inf_{\substack{I'\subseteq I\\ \text{finite}}} \prod_{i\in I'} k_i$$

for all families $(k_i)_{i\in I}$ in $\mathcal{K}$. Then, the infinitary operations in $\mathcal{K}$ must be monotone.

Proof. Given any two families $(k_i)_{i\in I}$ and $(\ell_i)_{i\in I}$ of elements in $\mathcal{K}$, we have that

$$\sum_{i\in I}(k_i+\ell_i) = \sup_{\substack{I'\subseteq I\\ \text{finite}}} \sum_{i\in I'} k_i+\ell_i \overset{(*)}{\geq} \sup_{\substack{I'\subseteq I\\ \text{finite}}} \sum_{i\in I'} k_i = \sum_{i\in I} k_i \text{ and}$$
$$\prod_{i\in I}(k_i+\ell_i) = \inf_{\substack{I'\subseteq I\\ \text{finite}}} \prod_{i\in I'} k_i+\ell_i \overset{(*)}{\geq} \inf_{\substack{I'\subseteq I\\ \text{finite}}} \prod_{i\in I'} k_i = \prod_{i\in I} k_i.$$

The steps which are marked by $(*)$ follow from the monotonicity of the finite operations in $\mathcal{K}$. $\square$

Hence, the infinitary operations in $\mathbb{V}$ must be monotone and the modified isomorphism lemma can be applied to infinite $\mathbb{V}$-interpretations in order to prove the elementary equivalence of π_A^7 and π_B^7.

Theorem 3.19. There are $\mathbb{V}$-interpretations π_A, π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$.

Proof. Recall the $\mathbb{V}$-interpretations π_A^7 and π_B^7. The universes of π_A^7 and π_B^7 consist of elements a_i and b_i where $i \in \mathbb{N}_{>0}$. The valuations are defined by $\pi_A^7(Ra_i) = \pi_B^7(Rb_i) = 1$ if i is even, while $\pi_A^7(Ra_i) = 0.1$ and $\pi_B^7(Rb_i) = 0.01$ for any odd i.

π_A^7 :

A	R	$\neg R$
a_1	0.1	0
a_2	1	0
a_3	0.1	0
a_4	1	0
$\vdots$	$\vdots$	$\vdots$

π_B^7 :

B	R	$\neg R$
b_1	0.01	0
b_2	1	0
b_3	0.01	0
b_4	1	0
$\vdots$	$\vdots$	$\vdots$

In order to prove $\pi_A^7 \equiv \pi_B^7$, we first show that for all formulae $\varphi(\bar{x}) \in \mathrm{FO}(\{R\})$, $\bar{a} = (a_{i_1}, \ldots, a_{i_n}) \in A^n$ and $\bar{b} = (b_{j_1}, \ldots, b_{j_n}) \in B^n$ with i_ℓ and j_ℓ even for $1 \leq \ell \leq n$, it holds that

$$\{\pi_A^7[\![\varphi(\bar{a})]\!], \pi_B^7[\![\varphi(\bar{b})]\!]\} \subseteq \{0,1\}.$$

The claim can be shown by induction on the structure of $\varphi(\bar{x})$. We only prove the claim for π_A^7, as the exact same reasoning can be transferred to π_B^7. Fix some $\bar{a} = (a_{i_1}, \ldots, a_{i_n}) \in A^n$ such that i_ℓ is even, i.e., $\pi_A^7(Ra_{i_\ell}) = 1$ for $1 \leq \ell \leq n$.

Case 1. For all literals $\varphi(\bar{x})$, the claim follows immediately from the definition of π_A^7, as

$$\begin{aligned}
\pi_A^7[\![a_{i_\ell} = a_{i_\ell}]\!] = \pi_A^7[\![a_{i_\ell} \neq a_{i'_\ell}]\!] = \pi_A^7(Ra_{i_\ell}) = 1 \text{ and} \\
\pi_A^7[\![a_{i_\ell} \neq a_{i_\ell}]\!] = \pi_A^7[\![a_{i_\ell} = a_{i'_\ell}]\!] = \pi_A^7(\neg Ra_{i_\ell}) = 0
\end{aligned}$$

for all $1 \leq \ell < \ell' \leq n$, which implies the base case.

Case 2. If $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$ where $\circ \in \{\vee, \wedge\}$, we can apply the induction hypothesis to both $\psi(\bar{x})$ and $\vartheta(\bar{x})$, which yields

$$\{\pi_A^7[\![\psi(\bar{a})]\!], \pi_A^7[\![\vartheta(\bar{a})]\!]\} \subseteq \{0,1\},$$

and infer that

$$\begin{aligned}
\pi_A^7[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] = \max(\pi_A^7[\![\psi(\bar{a})]\!], \pi_A^7[\![\vartheta(\bar{a})]\!]) \in \{0,1\} \text{ and} \\
\pi_A^7[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!] = \pi_A^7[\![\psi(\bar{a})]\!] \cdot \pi_A^7[\![\vartheta(\bar{a})]\!] \in \{0,1\}.
\end{aligned}$$

Case 3. If $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$, applying the induction hypothesis to $\psi(\bar{x}, x)$ implies that for all $a \in A$ with $\pi_A^7(Ra) = 1$, it holds that $\pi_A^7[\![\psi(\bar{a}, a)]\!] \in \{0,1\}$. If there is some $a \in A$ such that $\pi_A^7[\![\psi(\bar{a}, a)]\!] = 1$, it immediately follows that

$$\pi_A^7[\![\varphi(\bar{a})]\!] = \sup_{a \in A} \pi_A^7[\![\psi(\bar{a}, a)]\!] = 1.$$

Hence, it remains to show the claim for the case $\pi_A^7[\![\psi(\bar{a}, a)]\!] = 0$ for all $a \in A$ with $\pi_A^7(Ra) = 1$. Fix some $a \in A$ with $\pi_A(Ra) = 1$ such that $a \notin \{a_{i_1}, \ldots, a_{i_\ell}\}$. For each $a' \in A$ with $\pi_A(Ra) = 0.1$ it holds that $(\pi_A, \bar{a}, a')$ is $\leq$-isomorphic to $(\pi_A, \bar{a}, a)$. Using lemma 3.17, we can conclude that $\pi_A^7[\![\psi(\bar{a}, a')]\!] \leq \pi_A^7[\![\psi(\bar{a}, a)]\!] = 0$ and hence

$$\pi_A^7[\![\varphi(\bar{a})]\!] = \max_{a \in A} \pi_A^7[\![\psi(\bar{a}, a)]\!] = 0.$$

Case 4. If $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$, then for all $a \in A$ with $\pi_A^7(Ra) = 1$ it holds that $\pi_A^7[\![\psi(\bar{a}, a)]\!] \in \{0,1\}$ by induction hypothesis. If there is some $a \in A$ such that $\pi_A^7[\![\psi(\bar{a}, a)]\!] = 0$, it immediately follows that

$$\pi_A^7[\![\varphi(\bar{a})]\!] = \prod_{a \in A} \pi_A^7[\![\psi(\bar{a}, a)]\!] = 0.$$

Therefore it remains to show the claim for the case $\pi^7_A[\![\psi(\bar{a},a)]\!] = 1$ for all $a \in A$ with $\pi^7_A(Ra) = 1$. We observe that for all $a, a' \in A$ with $a \neq a_{i_\ell}$ and $a' \neq a_{i_\ell}$ for $1 \leq \ell \leq n$ as well as $\pi^7_A(Ra) = \pi^7_A(Ra')$ it holds that $(\pi^7_A, \bar{a}, a) \cong (\pi^7_A, \bar{a}, a')$. Hence, if there was some $a \in A$ with $\pi^7_A(Ra) = 0.1$ such that $\pi^7_A[\![\psi(\bar{a},a)]\!] = k$ for some $k \in [0,1)_{\mathbb{R}}$, then $\pi^7_A[\![\psi(\bar{a},a)]\!] = k$ would hold for all $a \in A$ with $\pi^7_A(Ra) = 0.1$, which implies

$$\pi^7_A[\![\varphi(\bar{a})]\!] = \prod_{a \in A} \pi^7_A[\![\psi(\bar{a},a)]\!] = 0.$$

Otherwise, we have that $\pi^7_A[\![\psi(\bar{a},a)]\!] = 1$ for all $a \in A$, thus $\pi^7_A[\![\varphi(\bar{a})]\!] = 1$, which completes the induction.

In particular, we showed that $\{\pi^7_A[\![\varphi]\!], \pi^7_B[\![\varphi]\!]\} \subseteq \{0,1\}$ holds for all sentences $\varphi \in \mathrm{FO}(\{R\})$. The mapping $h\colon \mathbb{V} \to \mathbb{V}$ defined by $k \mapsto k^2$ is an endomorphism which is compatible with the infinitary operations, because

(1) $h(0) = 0^2 = 0$ and $h(1) = 1^2 = 1$,

(2) $h(\max(k,\ell)) = \max(k,\ell)^2 = \max(k^2,\ell^2) = \max(h(k), h(\ell))$,

(3) $h(k \cdot \ell) = (k \cdot \ell)^2 = k^2 \cdot \ell^2$,

(4) $(\sum_{i \in I} k_i)^2 = (\sup_{i \in I} k_i)^2 = \sup_{i \in I} k_i^2 = \sum_{i \in I} k_i^2$ and

(5) $(\prod_{i \in I} k_i)^2 = \prod_{i \in I} k_i^2$.

Note that (4) is due to the fact that $k \leq \ell$ if, and only if, $k^2 \leq \ell^2$ for all $k, \ell \in \mathbb{V}$ and (5) follows from partition invariance of infinitary multiplication in $\mathbb{V}$. Towards a contradiction, suppose there was a sentence φ separating π^7_A and π^7_B. Using the claim above, it would either hold that $\pi^7_A[\![\varphi]\!] = 0$ and $\pi^7_B[\![\varphi]\!] = 1$, or that $\pi^7_A[\![\varphi]\!] = 1$ and $\pi^7_B[\![\varphi]\!] = 0$. Due to the fundamental property, this would imply

$$\begin{aligned} h \circ \pi^7_A[\![\varphi]\!] &= h(\pi^7_A[\![\varphi]\!]) = 0^2 = 0 \neq 1 = \pi^7_B[\![\varphi]\!] \text{ or} \\ h \circ \pi^7_A[\![\varphi]\!] &= h(\pi^7_A[\![\varphi]\!]) = 1^2 = 1 \neq 0 = \pi^7_B[\![\varphi]\!]. \end{aligned}$$

But it holds that $(h \circ \pi^7_A) \cong \pi^7_B$, which yields a contradiction. Hence, it must hold that $\pi^7_A \equiv \pi^7_B$. $\square$

Notice that the elementary equivalence of π^7_A and π^7_B also shows that elementary equivalence does not imply isomorphism for infinite $\mathbb{V}$-interpretations, not even if the vocabulary consists of a single unary relation symbol.

Soundness of G_m and G. After analyzing in what cases the (m-)separability of $\mathbb{V}$-interpretations can be inferred if Spoiler wins the game G_m or G, we will now move on to the transferability of the games as a method to show m-equivalence and elementary equivalence. Unlike min-max semirings, the Viterbi semiring is not multiplicatively idempotent, which is why m-equivalence cannot be inferred from a winning strategy for Duplicator in G_m. Due to the absence of multiplicative idempotence, Spoiler might not be able to prove in m turns that he knows a separating formula of quantifier rank m. However, it might be sufficient to permit a fixed larger number of turns, so it is still possible that elementary equivalence is implied by a winning strategy of Duplicator in the game G. For finite $\mathbb{V}$-interpretations this implication is clearly true, as the winning of Duplicator already implies isomorphism in this case. According to the isomorphism lemma, this ensures elementary equivalence of the $\mathbb{V}$-interpretations. Certainly, this reasoning cannot be transferred to infinite $\mathbb{V}$-interpretations. Therefore, the remainder of this section will be concerned with the question of whether elementary equivalence is implied by the winning of Duplicator in the game G on infinite $\mathbb{V}$-interpretations.

First, we discuss in general how a winning strategy for Spoiler in the game G could be constructed based on a separating formula and derive what problems occur thereby. For any semiring $\mathcal{K}$ where the infinitary operations are defined via the supremum or infimum of the finite subsum and subproducts, as it is the case for $\mathbb{V}$, it holds that

$$\sum_{i\in I} k = \sum_{j\in J} k \text{ and } \prod_{i\in I} k = \prod_{j\in J} k$$

for all infinite index sets I,J and $k \in \mathcal{K}$. With invariance under partitions, this yields for all families $(k_i)_{i\in I}$ and $(\ell_j)_{j\in J}$ such that

$$|\{i \in I\colon k_i = k\}| = |\{j \in J\colon \ell_j = k\}| \text{ or}$$
$$|\{i \in I\colon k_i = k\}| \geq \omega \ \text{ and } \ |\{j \in J\colon \ell_j = k\}| \geq \omega$$

for all $k \in \mathcal{K}$, it has to hold that

$$\sum_{i\in I} k_i = \sum_{j\in J} \ell_j \text{ and } \prod_{i\in I} k_i = \prod_{j\in J} \ell_j.$$

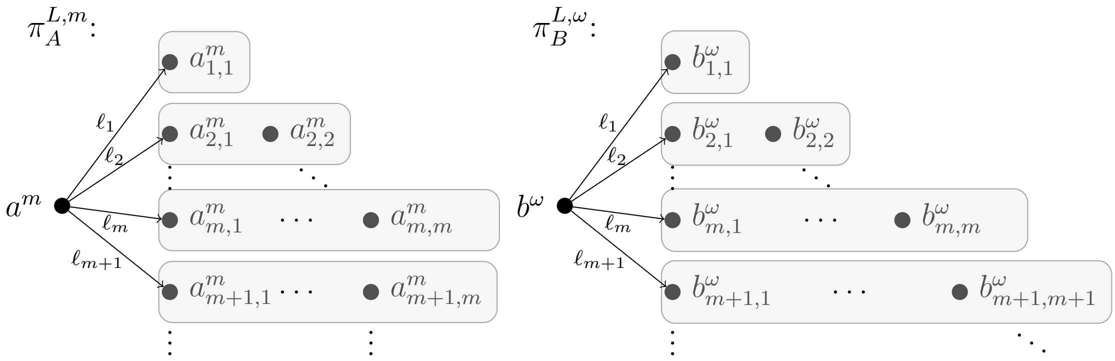
Note that in the Viterbi semiring, the equality of the sums is already implied by the equality of sets of summands due to idempotence. But the implication above follows from the definition of the infinitary operations only and thus holds in any semiring with analogous infinitary operations. The contraposition yields that for any formula $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ separating $\mathcal{K}$-interpretations $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$, there is some $k \in \mathcal{K}$ and a natural number $n \in \mathbb{N}$ such that

$$|\{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}| = n \text{ and } |\{b \in B\colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}| > n \text{ or}$$
$$|\{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}| > n \text{ and } |\{b \in B\colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}| = n.$$

Hence, Spoiler can make sure that, after finitely many steps, some pair (a_i, b_i) has been chosen such that $\pi_A[\![\psi(\bar{a}, a_i)]\!] \neq \pi_B[\![\psi(\bar{b}, b_i)]\!]$, i.e., any quantifier can be eliminated within finitely many turns. For sentences $\varphi = Q_1 x \psi(x)$, the number of moves needed to eliminate the outermost quantifier and ensure the existence of a pair (a_i, b_i) with $\pi_A[\![\psi(a_i)]\!] \neq \pi_B[\![\psi(b_i)]\!]$ within any reachable position only depends on π_A, π_B and φ, whereas this is not sufficient for nested quantifiers. Suppose, for example, that $\psi(x)$ has the form $\psi(x) = Q_2 y \vartheta(x, y)$. Spoiler has a strategy to pick finitely many elements such that the resulting position contains at least one pair (a_j, b_j) with $\pi_A[\![\vartheta(a_i, a_j)]\!] \neq \pi_B[\![\vartheta(b_i, b_j)]\!]$. Which elements have to be chosen by Spoiler and thus, the number of required elements to eliminate the inner quantifier may vary for different instantiations of the free variable x, that is, for different answers of Duplicator in the first turn. In order to construct a winning strategy for Spoiler in the game G, there must be a maximum number of elements Spoiler needs to pick in order to prove the separability of the $\mathcal{K}$-interpretations. But in the case of infinite universes, infinitely many different answers of Duplicator are possible. Hence, the number of turns required to eliminate a nested quantifier might be unbounded with respect to the instantiations of the variables referring to an outer quantifier.

This observation can be formulated in terms of the generalization G_α of the Ehrenfeucht-Fraïssé game to arbitrary ordinals α, which extends the classical game according to the following rule. If α is a limit ordinal, then Spoiler chooses an arbitrary ordinal $\beta < \alpha$ and the play proceeds according to G_β. Otherwise, $\alpha = \beta + 1$ is a successor ordinal, the players draw elements as in the original game and continue with the game G_β. For each semiring with infinitary operations defined via supremum or infimum as described, Spoiler has a winning strategy for the game $G_{\omega \cdot m}$ if there is a separating formula of quantifier rank m. In particular, in any of these semirings, 1-equivalence can be inferred from a winning strategy for Duplicator in the game $G_\omega = G$. Whether a winning strategy for Duplicator in G suffices to ensure m-equivalence for larger m remains to be shown.

The Viterbi semiring illustrates that the potential unboundedness of the number of required turns actually poses a problem and causes the winning of Duplicator in G not to imply elementary equivalence in general. In order to prove this claim, we construct separable $\mathbb{V}$-interpretations on which Duplicator wins the game G. We have already seen that it is not possible to construct a counterexample which can be separated with quantifier rank 1, which is why we construct $\mathbb{V}$-interpretations over $\tau = \{E\}$ where E is a binary relation symbol. The construction relies on the observation that if only m turns remain in the game G, then the node a^m depicted in the subsequent figure is an appropriate duplicate of the node b^ω. More precisely, for any sequence $L = (\ell_i)_{i \geq 1}$ of elements in $\mathbb{V}$, Duplicator wins the game $G_m(\pi_A^{L,m}, \pi_B^{L,\omega})$.



Besides the node a^m, the universe A contains elements $a^m_{i,j}$ for all $i, j \in \mathbb{N}_{>0}$ such that $j \leq \min(i, m)$, whereas B consists of b^ω and elements $b^m_{i,j}$ for all $i, j \in \mathbb{N}_{>0}$ with $j \leq i$. The gray boxes are supposed to illustrate that $\pi_A^{L,m}(Ea^m a^m_{i,j}) = \ell_i$ and $\pi_B^{L,\omega}(Eb^\omega b^\omega_{i,j}) = \ell_i$ for all j. All remaining unnegated $\{E\}$-literals over A and B are valuated with 0, while their negations are assigned 1. Hence, $\pi_A^{L,m}$ and $\pi_B^{L,\omega}$ can be regarded as trees with root a^m and b^ω, correspondingly.

For every $i \geq 1$, there are exactly as many outgoing edges from a^m which are valuated with ℓ_i as there are from b^ω, or there are at least m outgoing edges valuated with ℓ_i from both a^m and b^ω. Hence, we can partition A and B into sets $\{A_i : i \in \mathbb{N}\}$ and $\{B_i : i \in \mathbb{N}\}$ such that $A_0 = \{a^m\}$ and $B_0 = \{b^\omega\}$, while $A_i = \{a^m_{i,j} : j \leq \min(i, m)\}$ and $B_i = \{b^\omega_{i,j} : j \leq i\}$ for $i \geq 1$ and obtain that $|A_i| = |B_i|$, or both $|A_i| \geq m$ and $|B_i| \geq m$ for each $i \in \mathbb{N}$. Duplicator wins the game $G_m(\pi_A^{L,m}, \pi_B^{L,\omega})$ by answering each $a \in A_i$ with some arbitrary $b \in B_i$ and vice versa such that equalities with regard to the previous choices are respected. This is possible for m turns and the resulting position must induce a local isomorphism.

However, for an appropriate sequence $L = (\ell_i)_{i \geq 1}$ of edge labels, $\pi_A^{L,m}$ and $\pi_B^{L,\omega}$ can be separated with a formula of quantifier rank 2. To prove this, we define values $\ell_i \in \mathbb{V}$ such that different valuations are obtained when multiplying over all outgoing edges from a^m compared to the product of edges valuations from b^ω. In this way, we aim to construct distinct $\mathbb{V}$-interpretations which are "sufficiently similar" but can still be separated.

Lemma 3.20. There is a family $(\ell_i)_{i \geq 1}$ of elements in $\mathbb{V}$ such that

$$0 < \prod_{i \in \mathbb{N}_{>0}} \ell_i^i < \prod_{i \in \mathbb{N}_{>0}} \ell_i^{\min(i,m)}$$

for all $m \in \mathbb{N}_{>0}$.

Proof. We prove the claim for $(\ell_i)_{i\geq 1}$ where $\ell_i := \exp\bigl(-\frac{1}{2^i \cdot i}\bigr)$. Note that for each $x \in [-1/2, 0]_{\mathbb{R}}$, we have that $\exp(x) \in [0,1]_{\mathbb{R}}$. Hence, $(\ell_i)_{i\geq 1}$ is a family of elements in $\mathbb{V}$. Using the fact that $\prod^{\mathbb{V}}_{i\in I} \exp(\ell_i) = \exp(\sum^{\mathbb{R}}_{i\in I}(\ell_i))$ for all $(\ell_i)_{i\in I}$ such that $\exp(\ell_i) \in \mathbb{V}$ and the convergence of the geometrical series we obtain that

$$\begin{aligned}
\prod_{i\in\mathbb{N}_{>0}} \ell_i^i &= \prod_{i\in\mathbb{N}_{>0}} \exp\Bigl(-\frac{1}{2^i \cdot i}\Bigr)^i \\
&= \prod_{i\in\mathbb{N}_{>0}} \exp\Bigl(-\frac{1}{2^i}\Bigr) \\
&= \exp\Bigl(\sum^{\mathbb{R}}_{i\in\mathbb{N}_{>0}} \Bigl(-\frac{1}{2^i}\Bigr)\Bigr) \\
&= \exp\Bigl(-\Bigl(\sum^{\mathbb{R}}_{i\in\mathbb{N}} \frac{1}{2^i}\Bigr) + 1\Bigr) \\
&= \exp(-1) > 0.
\end{aligned}$$

Further, for all $m \in \mathbb{N}_{>0}$ it holds that

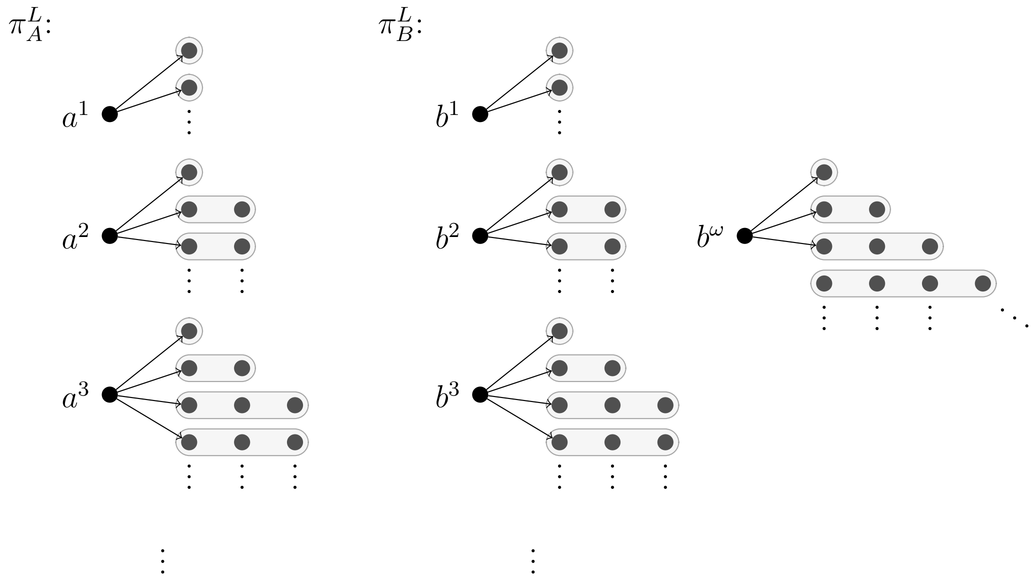
$$\underbrace{\prod_{i\in\mathbb{N}_{>0}} \ell_i^i}_{>0} = \prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)} \cdot \underbrace{\prod_{\substack{i\in\mathbb{N}_{>0}:\\ i>m}} \ell_i^{i-m}}_{<1} < \prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)},$$

which implies the claim. □

Consequently, there is a sequence L such that π_A^L and π_B^L can be separated by $\varphi = \exists x\psi(x)$ with $\psi(x) = \forall y(x = y \vee Exy)$, as

$$\pi_A^{L,m}[\![\varphi]\!] = \pi_A^{L,m}[\![\psi(a^m)]\!] = \prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)} \neq \prod_{i\in\mathbb{N}_{>0}} \ell_i^i = \pi_B^{L,\omega}[\![\psi(b^\omega)]\!] = \pi_B^{L,\omega}[\![\varphi]\!].$$

So, for arbitrarily large $m \in \mathbb{N}_{>0}$, there are 2-separable $\mathbb{V}$-interpretations $\pi_A^{L,m}$ and $\pi_B^{L,\omega}$ such that Duplicator wins $G_m(\pi_A^{L,m}, \pi_B^{L,\omega})$. Based on $\pi_A^{L,m}$ and $\pi_B^{L,\omega}$, we aim to construct $\mathbb{V}$-interpretations π_A^L and π_B^L which preserve the separability but on which Duplicator wins G. In order to account for any possible number of turns, π_A^L contains all $\pi_A^{L,m}$ as disjoint subgraphs. Since distinct nodes a^n and a^m cannot be duplicated with the same node b^ω, the $\mathbb{V}$-interpretation π_B^L does not only contain $\pi_B^{L,\omega}$ as a subgraph, but we additionally include a copy of each $\pi_A^{L,m}$. In order to be able to multiply over all outgoing edges of the root nodes without obtaining 0, we add certain additional edges such that $\pi_A^L(Ea^m a^n) = \pi_A^L(Ea^m a^n_{i,j}) = 1$ for $n \neq m$, and the analogous for π_B^L.



The definition of π_A^L and π_B^L ensures that Duplicator wins G, but the sentence $\exists x\psi(x)$ with $\psi(x) = \forall y(x = y \vee Exy)$, which separates each $\pi_A^{L,m}$ and $\pi_B^{L,\omega}$, does not lead to distinct valuations in π_A^L and π_B^L. Due to lemma 3.20, we obtain for all $m \in \mathbb{N}_{>0}$ that

$$\pi_A[\![\psi(a^m)]\!] = \pi_B[\![\psi(b^m)]\!] = \prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)} > \prod_{i\in\mathbb{N}_{>0}} \ell_i^i = \pi_B[\![\psi(b^\omega)]\!].$$

Hence, the additional node b^ω in π_B^L does not affect the valuation of $\exists x\psi(x)$, which is why we make a further modification in order to obtain the final counterexample. Recall the property which ensures the winning of Duplicator in $G_m(\pi_A^{L,m}, \pi_B^{L,\omega})$. The crucial observation was that for each $i \geq 1$, there are equally many edges labeled with ℓ_i from a^m and b^ω, or at least m from both a^m and b^ω. We defined π_A^L and π_B^L such that from each a^m and b^m, there are exactly m edges which are labeled with ℓ_m. Adding more such edges by incorporating additional nodes $a^m_{m,m+j}$ and $b^m_{m,m+j}$ would not violate this property. Hence, under the assumption that the additional nodes preserve the remaining structure of π_A^L and π_B^L, Duplicator still wins the game G on the resulting $\mathbb{V}$-interpretations. By adding sufficiently many such edges, we can ensure that the product over all outgoing edges from a^m is smaller than $\exp(-1)$ and decreases for growing $m \in \mathbb{N}_{>0}$. This causes the node b^ω to determine the valuation of $\exists x\psi(x)$ such that the resulting $\mathbb{V}$-interpretations can be separated from each other. Making use of these ideas and observations, we are ready to formally construct the counterexample.

Theorem 3.21. There are $\mathbb{V}$-interpretations π_A and π_B such that Duplicator wins $G(\pi_A, \pi_B)$ and $\pi_A \not\equiv_2 \pi_B$.

Proof. Let $(\ell_i)_{i\geq 1}$ be defined by $\ell_i := \exp(\frac{1}{2^i i})$ and let ℓ_∞^i denote $\prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)}$ for each $i \geq 1$. We inductively define a function $f\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ as follows. Let $f(1)$ be the smallest number such that

$$\ell_\infty^1 \cdot \ell_1^{f(1)} < \exp(-1).$$

For any $i > 1$, we define $f(i)$ as the minimum number yielding

$$\ell_\infty^i \cdot \ell_i^{f(i)} \leq \ell_\infty^{i-1} \cdot \ell_{i-1}^{f(i-1)}.$$

Since $0 < \ell_i < 1$ for all $i \geq 1$, f is well-defined. Hence, we obtain a chain $\ell_\infty^1 \cdot \ell_1^{f(1)} \geq \ell_\infty^2 \cdot \ell_2^{f(2)} \geq \dots$ which is strictly upper bounded by $\exp(-1)$. Based on f and $(\ell_i)_{i\geq 1}$, we construct $\mathbb{V}$-interpretations $\pi_A^{L,f}$ and $\pi_B^{L,f}$ over the vocabulary $\tau = \{E\}$ consisting of a binary relation symbol. The universes A and B contain the following elements.

$$\begin{aligned} A &= \{a^m\colon m \in \mathbb{N}_{>0}\} \cup \{a_{i,j}^m\colon m \in \mathbb{N}_{>0}, j \leq \min(i,m)\} \cup \{a_{m,m+j}^m\colon j \leq f(m)\} \\ B &= \{b^m\colon m \in \mathbb{N}_{>0}\} \cup \{b_{i,j}^m\colon m \in \mathbb{N}_{>0}, j \leq \min(i,m)\} \cup \{b_{m,m+j}^m\colon j \leq f(m)\} \\ &\quad \cup \{b^\omega\} \cup \{b_{ij}^\omega\colon j \leq i\} \end{aligned}$$

The valuations in $\pi_A^{L,f}$ and $\pi_B^{L,f}$ are defined according to the following rules, which apply to all $m, n, i, j \in \mathbb{N}_{>0}$ with $m \neq n$ such that the respective nodes are contained in A or B.

- $\pi_A^{L,f}(Ea^m a_{i,j}^m) = \pi_B^{L,f}(Eb^m b_{i,j}^m) = \pi_B^{L,f}(Eb^\omega b_{i,j}^\omega) = \ell_i$
- $\pi_A^{L,f}(Ea^m a_{i,j}^n) = \pi_B^{L,f}(Eb^m b_{i,j}^n) = \pi_B^{L,f}(Eb^\omega b_{i,j}^m) = \pi_B^{L,f}(Eb^m b_{i,j}^\omega) = 1$
- $\pi_A^{L,f}(Ea^m a^n) = \pi_B^{L,f}(Eb^m b^n) = \pi_B^{L,f}(Eb^\omega b^m) = \pi_B^{L,f}(Eb^m b^\omega) = 1$

Further, the negations of the τ-literals over A and B defined above are valuated with 0. All remaining unnegated τ-literals over A and B are valuated with 0 and their negations with 1.

Using invariance under partitions of infinitary multiplication, we obtain the following valuations of the formula $\psi(x) = \forall y(x = y \vee Exy)$.

- $\pi_A^{L,f}[\![\psi(a_{i,j}^m)]\!] = \pi_B^{L,f}[\![\psi(b_{i,j}^m)]\!] = \pi_B^{L,f}[\![\psi(b_{i,j}^\omega)]\!] = 0$
- $\pi_A^{L,f}[\![\psi(a^m)]\!] = \pi_B^{L,f}[\![\psi(b^m)]\!] = \big(\prod_{i\in\mathbb{N}_{>0}} \ell_i^{\min(i,m)}\big) \cdot \ell_m^{f(m)} = \ell_\infty^m \cdot \ell_m^{f(m)}$
- $\pi_B^{L,f}[\![\psi(b^\omega)]\!] = \prod_{i\in\mathbb{N}_{>0}} \ell_i^i = \exp(-1)$

Since we constructed f such that $\ell^1_\infty \cdot \ell^{f(1)}_1 \geq \ell^2_\infty \cdot \ell^{f(2)}_2 \geq \dots$ is strictly upper bounded by $\exp(-1)$, this implies

$$\pi^{L,f}_A [\![\exists x \psi(x)]\!] = \ell^1_\infty \cdot \ell^{f(1)}_1 < \exp(-1) = \pi^{L,f}_B [\![\exists x \psi(x)]\!],$$

hence $\pi^{L,f}_A \not\equiv_2 \pi^{L,f}_B$.

In order to construct a winning strategy for Duplicator in the game $G(\pi^{L,f}_A, \pi^{L,f}_B)$, let $\mathcal{P}_A := \{A^n_i : n \in \mathbb{N}_{>0}, i \in \mathbb{N}\}$ and $\mathcal{P}_B := \{B^n_i : n \in \mathbb{N}_{>0} \cup \{\omega\}, i \in \mathbb{N}\}$ be partitions of A and B such that $A^n_0 = \{a^n\}$ and A^n_i for $i \geq 1$ contains all elements a^n_{ij} in A. Analogously, $B^n_0 = \{b^n\}$ and B^n_i consists of all elements b^n_{ij} where $n \in \mathbb{N}_{>0} \cup \{\omega\}$. Based on the number of turns m Spoiler chooses in the game $G(\pi^{L,f}_A, \pi^{L,f}_B)$, we define a bijection $g_m : \mathcal{P}_A \to \mathcal{P}_B$ as follows.

$$g_m(A^n_i) := \begin{cases} B^n_i, & n < m \\ B^\omega_i, & n = m \\ B^{n-1}_i, & n > m \end{cases}$$

Duplicator wins the game $G_m(\pi^{L,f}_A, \pi^{L,f}_B)$ by answering any element in A^n_i with an arbitrary element in $g_m(A^n_i)$ and every element in B^n_i with any element in $g^{-1}_m(B^n_i)$, merely making sure that (in)equalities with regard to the elements that have already been chosen are respected. This is possible, because for each A^n_i we have that $|A^n_i| = |g_m(A^n_i)|$ or that $|A^n_i| \geq m$ and $|g_m(A^n_i)| \geq m$. $\square$

3.1.3 The Łukasiewicz semiring $\mathbb{L}$

The Łukasiewicz semiring $\mathbb{L} = ([0,1]_\mathbb{R}, \max, \odot, 0, 1)$ relies on the same set of elements and uses the same addition as the Viterbi semiring and is thus also idempotent and naturally ordered by the usual order on $[0,1]_\mathbb{R}$. Multiplication in $\mathbb{L}$ is defined by $k \odot \ell = \max(k + \ell - 1, 0)$, hence $\mathbb{L}$ is not multiplicatively idempotent but absorptive. As multiplication decreases elements, we define the infinitary operations analogous to $\mathbb{V}$. While it immediately follows from theorem 3.5 that m-equivalence of $\mathbb{L}$-interpretations does not follow from the game G_m in general, we obtain the same result for the game G and elementary equivalence. In order to prove this, we make use of the isomorphic variant $\mathbb{D} = ([0,1]_\mathbb{R}, \min, \oplus, 1, 0)$ of $\mathbb{L}$ where $k \oplus \ell = \min(a + b, 1)$ and construct a counterexample which relies on the same construction as the counterexample for $\mathbb{V}$ but uses a different sequence $(\ell_i)_{i \geq 1}$.

Proposition 3.22. There are $\mathbb{D}$-interpretations π_A and π_B such that $\pi_A \not\equiv_2 \pi_B$, while Duplicator wins the game $G(\pi_A, \pi_B)$

Proof. For each $i \geq 1$, let $\ell_i := \frac{1}{2^{i+1} \cdot i}$. With respect to the usual addition and multiplication on $\mathbb{R}$, we obtain that

$$\begin{aligned}\sum_{i \in \mathbb{N}_{>0}} i \cdot \ell_i &= \sum_{i \in \mathbb{N}_{>0}} i \cdot \frac{1}{2^{i+1} i} \\ &= \sum_{i \in \mathbb{N}_{>0}} \frac{1}{2^{i+1}} \\ &= \Big(\sum_{i \in \mathbb{N}} \frac{1}{2^{i+1}} \Big) - \frac{1}{2^0} - \frac{1}{2^1} \\ &= 2 - 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Further, we can conclude for all $m \in \mathbb{N}_{>0}$ that

$$\sum_{i \in \mathbb{N}_{>0}} i \cdot \ell_i = \sum_{i \in \mathbb{N}_{>0}} \min(i,m) \cdot \ell_i + \underbrace{\sum_{\substack{i \in \mathbb{N}_{>0}: \\ i > m}} (m-i) \cdot \ell_i}_{>0} > \sum_{i \in \mathbb{N}_{>0}} \min(i,m) \cdot \ell_i.$$

Hence, every finite subsum of $\sum_{i \in \mathbb{N}_{>0}} i \cdot \ell_i$ and $\sum_{i \in \mathbb{N}_{>0}} \min(i,m) \cdot \ell_i$ for each $m \in \mathbb{N}_{>0}$ is upper bounded by 0.5 and thus leads to the same result as multiplication in $\mathbb{D}$, as 1 is not exceeded. Note that the natural order in $\mathbb{D}$ is inverted compared to the usual order on $[0,1]_{\mathbb{R}}$, because addition is defined by minimum. Hence, the infimum over a set of elements in $\mathbb{D}$ coincides with the supremum in $\mathbb{R}$ and we can infer that

$$0.5 = \prod_{i \in \mathbb{N}_{>0}}^{\mathbb{D}} \ell_i^i > \prod_{i \in \mathbb{N}_{>0}}^{\mathbb{D}} \ell_i^{\min(i,m)} =: \ell_\infty^m,$$

where the exponents refer to the repeated application of multiplication in $\mathbb{D}$, which corresponds to addition in $\mathbb{R}$.

Based on the sequence $(\ell_i)_{i \geq 1}$, we inductively define a function $f \colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$. Let $f(1)$ be minimal such that $\ell_\infty^1 \cdot^{\mathbb{D}} \ell_1^{f(1)} > 0.5$ and $f(i)$ be the smallest number such that

$$\ell_\infty^i \cdot^{\mathbb{D}} \ell_i^{f(i)} \geq \ell_\infty^{i-1} \cdot^{\mathbb{D}} \ell_{i-1}^{f(i-1)}$$

for $i \geq 1$. The function f is well-defined, since $\ell_i > 0$ for all $i \geq 1$. Hence, we obtain a non-decreasing sequence $\ell_\infty^1 \cdot^{\mathbb{D}} \ell_1^{f(1)} \leq \ell_\infty^2 \cdot^{\mathbb{D}} \ell_2^{f(2)} \leq \dots$ which is strictly lower bounded by 0.5.

We construct $\mathbb{D}$-interpretations $\pi_A^{L,f}$ and $\pi_B^{L,f}$ analogous to the $\mathbb{V}$-interpretations in the proof of theorem 3.21. It holds that $\pi_A^{L,f} \not\equiv_2 \pi_B^{L,f}$, since

$$\pi_A^{L,f}[\![\exists x \forall y (x = y \vee Exy)]\!] = \ell_\infty^1 \cdot^{\mathbb{D}} \ell_1^{f(1)} > 0.5 = \pi_B^{L,f}[\![\exists x \forall y (x = y \vee Exy)]\!].$$

But Duplicator wins the game $G(\pi_A^{L,f}, \pi_B^{L,f})$ as described in 3.21. $\square$

Despite the similarities of the Łukasiewicz semiring and the Viterbi semiring, elementary equivalence of finite $\mathbb{L}$-interpretations does not imply isomorphism, as it is the case for $\mathbb{V}$. This is mainly due to the fact that, in contrast to $\mathbb{V}$, $\mathbb{L}$ does not admit cancellation. In the counterexample given in [GM21], Spoiler already wins the game G_1, which illustrates that neither G_1 nor G is appropriate for proving m-equivalence or elementary equivalence of $\mathbb{L}$-interpretations, even if they are finite.

Proposition 3.23. There are finite $\mathbb{L}$-interpretations π_A and π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$.

It is noticeable that neither of the implications concerning the games G_m and G for capturing m-equivalence and elementary equivalence applies to the semirings $\mathbb{V}$ and $\mathbb{L}$. This is connected to the property that both $\mathbb{V}$ and $\mathbb{L}$ are idempotent but not multiplicatively idempotent. On the one hand, multiple occurrences of certain valuations affect the semantics of a universally quantified formula which causes Duplicator to win more often than desired. But at the same time, idempotence decreases the expressive power of the logic such that distinct valuations may lead to the winning of Spoiler, although they do not affect the valuation of any formula.

3.1.4 The Semiring $\mathbb{N}^\infty$

Unlike the semirings considered previously, the semiring $\mathbb{N}^\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ is not idempotent. It extends the natural semiring $\mathbb{N}$ by the element ∞, ensuring that infinitary operations are admitted. The operations on the additional element are defined by $n+\infty = \infty$ for all $n \in \mathbb{N}^\infty$ and $n \cdot \infty = \infty$ for all $n \in \mathbb{N}^\infty \setminus \{0\}$. Due to the absence of absorption, multiplication is not decreasing. In fact, multiplication is increasing on $\mathbb{N}^\infty \setminus \{0\}$, which is why we consider the supremum both of the finite subsums and subproducts as infinitary operations.

In the previous sections, we showed that the winning of Duplicator in G ensures elementary equivalence for min-max semirings, which are fully idempotent, whereas this is not true for $\mathbb{V}$ and $\mathbb{L}$. This observation suggests that full idempotence might not only characterize the class of semirings for which m-equivalence is implied by Duplicator winning G_m but also the semirings in which elementary equivalence follows from G. However, the semiring $\mathbb{N}^\infty$ illustrates that, although neither of the operations is idempotent, yet a winning strategy for Duplicator in G is sufficient for proving elementary equivalence. Hence, in the case of $\mathbb{N}^\infty$, applying the Ehrenfeucht-Fraïssé game to show equivalence fails at the relation between the number of turns and the quantifier rank only. This is mainly based on two observations. First, there is only one element corresponding to infinity, so a separating formula must always evaluate to a natural number in at least one of the $\mathbb{N}^\infty$-interpretations. Moreover, if a sum evaluates to a natural number k, then there can be at most k non-zero summands. Analogously, if we consider a product evaluating to $k > 0$, then there are at most k factors other than 1. This makes it possible for Spoiler to eliminate each quantifier in a separating formula by drawing at most k elements if the separating formula evaluates to a number smaller than k in at least one of the interpretations. So, in order to inductively construct Spoiler's winning strategy, we make use of an invariant stating that one of the valuations of the separating formula is kept small. Due to the fact that both addition and multiplication are increasing on $\mathbb{N}^\infty \setminus \{0\}$, this invariant can be propagated to the separating subformulae.

Theorem 3.24. Let π_A and π_B be $\mathbb{N}^\infty$-interpretations, $\bar{a} = (a_1, \dots, a_n) \in A^n$, $\bar{b} = (b_1, \dots, b_n) \in B^n$ and $k \in \mathbb{N}_{>0}$. If there is a separating formula $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) \leq m$ such that $\pi_A[\![\varphi(\bar{a})]\!] < k$ or $\pi_B[\![\varphi(\bar{b})]\!] < k$, then Spoiler wins $G_{k\cdot m}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Proof. We show the claim by induction on the structure of $\varphi(\bar{x})$ for all $\bar{a}$, $\bar{b}$ and k at the same time.

Case 1. In case $\varphi(\bar{x})$ is a literal, it follows immediately that $\sigma\colon a_i \mapsto b_i$ cannot be a local isomorphism and Spoiler wins the game $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) \leq m$, assume w.l.o.g. that $\pi_A[\![\varphi(\bar{a})]\!] < k$. Since addition is increasing in $\mathbb{N}^\infty$, it must hold that $\pi_A[\![\varphi_1(\bar{a})]\!] < k$ and $\pi_A[\![\varphi_2(\bar{a})]\!] < k$. Moreover, $\pi_A[\![\varphi(\bar{a})]\!] \neq \pi_B[\![\varphi(\bar{b})]\!]$ implies $\pi_A[\![\varphi_i(\bar{a})]\!] \neq \pi_B[\![\varphi_i(\bar{b})]\!]$ for some $i \in \{1, 2\}$. Applying the induction hypothesis to $\varphi_i(\bar{x})$ yields that Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 3. If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ such that $\mathrm{qr}(\varphi(\bar{x})) \leq m$, assume again that $\pi_A[\![\varphi(\bar{a})]\!] < k$. If $\pi_A[\![\varphi_1(\bar{a})]\!] < k$ and $\pi_A[\![\varphi_2(\bar{a})]\!] < k$, the reasoning from case 2 can be transferred. Otherwise, $\pi_A[\![\varphi_i(\bar{a})]\!] = 0$ must hold for some $i \in \{1, 2\}$. Since $\varphi(\bar{x})$ separates $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$ by assumption, we have that $\pi_B[\![\varphi_i(\bar{b})]\!] \neq 0$. Hence, $\varphi_i(\bar{x})$ is separating as well and the induction hypothesis can be applied, so Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 4. For $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$ such that $\mathrm{qr}(\varphi(\bar{x})) \leq m$, suppose w.l.o.g. that $\pi_A[\![\varphi(\bar{a})]\!] < \pi_B[\![\varphi(\bar{b})]\!]$ and let $A' := \{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] > 0\}$. It holds that $|A'| < k$, since $\pi_A[\![\varphi(\bar{a})]\!] < k$ by assumption. In the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$, Spoiler successively draws all elements $a \in A'$. If Duplicator manages to find for each $a \in A'$ a unique duplicate $b \in B$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] = \pi_B[\![\psi(\bar{b}, b)]\!]$, then there must be an $(|A'| + 1)$-th element in $b \in B$ with $\pi_B[\![\psi(\bar{b}, b)]\!] > 0$, because

$\pi_A[\![\varphi(\bar{a})]\!] < \pi_B[\![\varphi(\bar{b})]\!]$. Hence, if Duplicator was able to duplicate all previous choices, Spoiler additionally chooses such $b \in B$ afterwards. In both cases, the strategy results in a position $(a_1, \ldots, a_{n+\ell}, b_1, \ldots b_{n+\ell})$ where $\ell \leq k$ such that there is some $1 \leq i \leq \ell$ with $\pi_A[\![\psi(\bar{a}, a_{n+i})]\!] \neq \pi_B[\![\psi(\bar{b}, b_{n+i})]\!]$. Since $\pi_A[\![\varphi(\bar{a})]\!] < k$ by assumption, it holds that $\pi_A[\![\psi(\bar{a}, a)]\!] < k$ for all $a \in A$ and the induction hypothesis can be applied to $\psi(\bar{x}, x)$ with instantiations $(\bar{a}, a_{n+i})$ and $(\bar{b}, b_{n+i})$. We obtain that Spoiler wins the game $G_{k(m-1)}(\pi_A, \bar{a}, a_{n+i}, \pi_B, \bar{b}, b_{n+i})$. As $\ell \leq k$, this implies that Spoiler also wins the remaining subgame.

Case 5. If $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$ such that $\mathrm{qr}(\varphi(\bar{x})) \leq m$, again suppose that $\pi_A[\![\varphi(\bar{a})]\!] < \pi_B[\![\varphi(\bar{b})]\!]$. If $\pi_A[\![\varphi(\bar{a})]\!] \neq 0$, the reasoning is analogous to case 4, since multiplication is increasing on $\mathbb{N}^\infty \setminus \{0\}$. The only difference is that the set $A' := \{a \in A \colon \pi_A[\![\psi(\bar{a}, a)]\!] > 1\}$ has to be considered. Otherwise, there must be some $a \in A$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] = 0$, whereas $\pi_B[\![\psi(\bar{b}, b)]\!] \neq 0$ for all $b \in B$, as $\pi_B[\![\varphi(\bar{b})]\!] > 0$. In the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$, Spoiler picks this element $a \in A$ such that $\psi(\bar{x}, x)$ is valuated with 0 in π_A. For any possible answer $b \in B$, the formula $\psi(\bar{x}, x)$ separates $(\pi_A, \bar{a}, a)$ from $(\pi_B, \bar{b}, b)$ and $\pi_A[\![\psi(\bar{a}, a)]\!] = 0 < k$. Applying the induction hypothesis yields that Spoiler wins the game $G_{k(m-1)}(\pi_A, \bar{a}, \pi_B, \bar{b})$. As Spoiler only picked one element in his strategy for $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$, at least $k(m-1)$ moves are left and he wins the remaining subgame as well. □

Consequently, the number of turns required to eliminate the quantifiers within a separating sentence φ can be bounded by some $k \in \mathbb{N}$. Note that the bound k depends on φ and the $\mathbb{N}^\infty$-interpretations the game is played on, i.e., there is no $k \in \mathbb{N}$ such that for all $\mathbb{N}^\infty$-interpretations π_A and π_B, Duplicator winning $G_{km}(\pi_A, \pi_B)$ implies $\pi_A \equiv_m \pi_B$. For instance, we can associate with each $k \in \mathbb{N}_{>0}$ the $\mathbb{N}^\infty$-interpretations π_A^k and π_B^{k+1} over $\tau = \varnothing$ with universes $A = \{a_i \colon i \in \mathbb{N}, i < k\}$ and $B = \{b_i \colon i \in \mathbb{N}, i < k+1\}$. Then, Duplicator clearly wins $G_k(\pi_A^k, \pi_B^{k+1})$ for each $k \in \mathbb{N}_{>0}$ despite 1-separability, as $\pi_A^k[\![\exists x(x = x)]\!] = k \neq k+1 = \pi_B^{k+1}[\![\exists x(x = x)]\!]$. Yet, we can conclude that full idempotence is a sufficient but not a necessary condition for the existence of a winning strategy for Duplicator in G implying elementary equivalence.

Corollary 3.25. If Duplicator wins the game $G(\pi_A, \pi_B)$ on $\mathbb{N}^\infty$-interpretations π_A and π_B, then π_A and π_B must be elementarily equivalent.

On the other hand, it can be shown that semiring semantics lacks the ability to distinguish between infinite $\mathbb{N}^\infty$-interpretations, even if they do not share a single valuation greater than 1. For proving this claim, we make use of the semiring $\mathbb{N}_{\leq 2}$ which coincides with $\mathbb{N}^\infty$ but is cut off at 2. In order to argue via the fundamental property, we apply the following lemma to prove that $\mathbb{N}_{\leq 2}$ is a semiring.

Lemma 3.26. Let $\mathcal{K} = (K, +^{\mathcal{K}}, \cdot^{\mathcal{K}}, 0^{\mathcal{K}}, 1^{\mathcal{K}})$ and $\mathcal{L} = (L, +^{\mathcal{L}}, \cdot^{\mathcal{L}}, 0^{\mathcal{L}}, 1^{\mathcal{L}})$ be algebraic structures and $h\colon \mathcal{K} \to \mathcal{L}$ a surjective homomorphism between $\mathcal{K}$ and $\mathcal{L}$. If $\mathcal{K}$ is a commutative semiring and $0^{\mathcal{L}} \neq 1^{\mathcal{L}}$, then $\mathcal{L}$ must also be a commutative semiring.

Proof. Let $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$ and $k_1, k_2, k_3 \in \mathcal{K}$ arbitrary such that $h(k_i) = \ell_i$ for $i \in \{1, 2, 3\}$. $(L, +^{\mathcal{L}}, 0^{\mathcal{L}})$ is a commutative monoid, because

(1) $(\ell_1 +^{\mathcal{L}} \ell_2) +^{\mathcal{L}} \ell_3 = (h(k_1) +^{\mathcal{L}} h(k_2)) +^{\mathcal{L}} h(k_3) = h((k_1 +^{\mathcal{K}} k_2) +^{\mathcal{K}} k_3) = h(k_1 +^{\mathcal{K}} (k_2 +^{\mathcal{K}} k_3)) = h(k_1) +^{\mathcal{L}} (h(k_2) +^{\mathcal{L}} h(k_3)) = \ell_1 +^{\mathcal{L}} (\ell_2 +^{\mathcal{L}} \ell_3)$,

(2) $\ell_1 +^{\mathcal{L}} \ell_2 = h(k_1) +^{\mathcal{L}} h(k_2) = h(k_1 +^{\mathcal{K}} k_2) = h(k_2 +^{\mathcal{K}} k_1) = h(k_2) +^{\mathcal{L}} h(k_1) = \ell_2 +^{\mathcal{L}} \ell_1$,

(3) $\ell_1 +^{\mathcal{L}} 0^{\mathcal{L}} = h(k_1) +^{\mathcal{L}} h(0^{\mathcal{K}}) = h(k_1 +^{\mathcal{K}} 0^{\mathcal{K}}) = h(k_1) = \ell_1$.

The same reasoning can be transferred to $(L, \cdot^{\mathcal{L}}, 1^{\mathcal{L}})$, which has to be a commutative monoid as well. Moreover, $0^{\mathcal{L}}$ must be annihilating and $\cdot^{\mathcal{L}}$ distributes over $+^{\mathcal{L}}$, as

(1) $\ell_1 \cdot^{\mathcal{L}} 0^{\mathcal{L}} = h(k_1) \cdot^{\mathcal{L}} h(0^{\mathcal{K}}) = h(k_1 \cdot^{\mathcal{K}} 0^{\mathcal{K}}) = h(0^{\mathcal{K}}) = 0^{\mathcal{L}}$ and

(2) $\ell_1 \cdot^{\mathcal{L}} (\ell_2 +^{\mathcal{L}} \ell_3) = h(k_1) \cdot^{\mathcal{L}} (h(k_2) +^{\mathcal{L}} h(k_3)) = h(k_1 \cdot^{\mathcal{K}} (k_2 +^{\mathcal{K}} k_3)) = h(k_1 k_2 +^{\mathcal{K}} k_1 k_3) = h(k_1)h(k_2) +^{\mathcal{L}} h(k_2)h(k_3) = \ell_1 \ell_2 +^{\mathcal{L}} \ell_1 \ell_3$.

Hence, $\mathcal{L}$ must be a commutative semiring as well. □

Lemma 3.27. The algebraic structure $\mathbb{N}_{\leq 2} = (\{0, 1, 2\}, +^{\mathbb{N}_{\leq 2}}, \cdot^{\mathbb{N}_{\leq 2}}, 0, 1)$ with operations defined by $a +^{\mathbb{N}_{\leq 2}} b = \min(a +^{\mathbb{N}^\infty} b, 2)$ and $a \cdot^{\mathbb{N}_{\leq 2}} b = \min(a \cdot^{\mathbb{N}^\infty} b, 2)$ is a commutative semiring.

Proof. The mapping $h\colon \mathbb{N}^\infty \to \mathbb{N}_{\leq 2}$ defined by $h\colon n \mapsto \min(n, 2)$ is a homomorphism, because $h(0) = \min(0, 2) = 0$, $h(1) = \min(1, 2) = 1$ and

$$
\begin{aligned}
h(n \star^{\mathbb{N}^\infty} m) &= \min(n \star^{\mathbb{N}^\infty} m, 2) \\
&= \min(\min(n, 2) \star^{\mathbb{N}^\infty} \min(m, 2), 2) = h(n) \star^{\mathbb{N}_{\leq 2}} h(m)
\end{aligned}
$$

for $\star \in \{+, \cdot\}$. Since $\mathbb{N}^\infty$ is a commutative semiring, h is surjective and the neutral elements do not coincide in $\mathbb{N}_{\leq 2}$, we can apply lemma 3.26 and infer that $\mathbb{N}_{\leq 2}$ is a commutative semiring. □

The infinitary operations we associate with $\mathbb{N}_{\leq 2}$ are, analogous to those in $\mathbb{N}^\infty$, defined by the supremum of the finite subsums and subproducts. This definition causes the homomorphism h from $\mathbb{N}^\infty$ to $\mathbb{N}_{\leq 2}$ to be compatible with the infinitary operations and enables the application of the fundamental property to infinite $\mathbb{N}^\infty$-interpretations and h. A crucial property of $\mathbb{N}^\infty$ we can make use of in order to derive a method for proving elementary equivalence of infinite $\mathbb{N}^\infty$-interpretations is that infinite sums and products in $\mathbb{N}^\infty$ always evaluate to ∞ if infinitely many summands/factors are greater than 1. This allows us to collapse the precise valuations > 1 in $\mathbb{N}^\infty$-interpretations π_A and π_B by making sure that if a valuation $\pi_A[\![\psi(a)]\!] > 1$ or $\pi_B[\![\psi(b)]\!] > 1$ occurs, then $\pi_A[\![\psi(a')]\!] > 1$ and $\pi_B[\![\psi(b')]\!] > 1$ applies to infinitely many $a' \in A$ and $b' \in B$.

Proposition 3.28. Let π_A and π_B be $\mathbb{N}^\infty$-interpretations and $h\colon \mathbb{N}^\infty \to \mathbb{N}_{\geq 2}$ with $n \mapsto \min(n, 2)$. If the universes of π_A and π_B can be partitioned into infinite sets $\{A_i\colon i \in I\}$ and $\{B_i\colon i \in I\}$ such that for each $i \in I$

$$(h \circ \pi_A, a) \equiv (h \circ \pi_A, a') \equiv (h \circ \pi_B, b) \equiv (h \circ \pi_B, b')$$

for all $a, a' \in A_i$ and $b, b' \in B_i$, then it must hold that $\pi_A \equiv \pi_B$.

Proof. Let π_A, π_B and partitions $\{A_i\colon i \in I\}$ and $\{B_i\colon i \in I\}$ be given as above. Applying the fundamental property yields for each $i \in I$ that

(1) $\pi_A[\![\psi(a)]\!] = 0 \Leftrightarrow \pi_A[\![\psi(a')]\!] = 0 \Leftrightarrow \pi_B[\![\psi(b)]\!] = 0 \Leftrightarrow \pi_B[\![\psi(b')]\!] = 0$,

(2) $\pi_A[\![\psi(a)]\!] = 1 \Leftrightarrow \pi_A[\![\psi(a')]\!] = 1 \Leftrightarrow \pi_B[\![\psi(b)]\!] = 1 \Leftrightarrow \pi_B[\![\psi(b')]\!] = 1$ and

(3) $\pi_A[\![\psi(a)]\!] \geq 2 \Leftrightarrow \pi_A[\![\psi(a')]\!] \geq 2 \Leftrightarrow \pi_B[\![\psi(b)]\!] \geq 2 \Leftrightarrow \pi_B[\![\psi(b')]\!] \geq 2$

for all formulae $\psi(x) \in \mathrm{FO}(\tau)$ and all $a, a' \in A_i$ and $b, b' \in B_i$. This implies for each $i \in I$ and all formulae $\psi(x)$ that

$$\sum_{a \in A_i} \pi_A[\![\psi(a)]\!] = \sum_{b \in B_i} \pi_B[\![\psi(b)]\!] \quad \text{and} \quad \prod_{a \in A_i} \pi_A[\![\psi(a)]\!] = \prod_{b \in B_i} \pi_B[\![\psi(b)]\!],$$

since each A_i and B_i is infinite. Due to invariance under partitions, this implies $\pi_A[\![\varphi]\!] = \pi_B[\![\varphi]\!]$ for all sentences $\varphi = Qx\psi(x)$ with $Q \in \{\exists, \forall\}$. If π_A and π_B were not elementarily equivalent, they would be separable by a sentence of this form due to lemma 2.19 and the fact that each literal contains a variable, which must be quantified in a sentence. Hence, it must hold that $\pi_A \equiv \pi_B$. $\square$

We apply this method in order to show that $\mathbb{N}^\infty$-interpretations can be elementarily equivalent, although Spoiler wins the game G, or even G_1. For instance, we can fix arbitrary infinite $\mathbb{N}^\infty$-interpretations π_A and π_B over $\tau = \{R\}$ where R is a unary relation symbol such that $\pi_A(Ra) \geq 2$, $\pi_B(Rb) \geq 2$ and $\pi_A(\neg Ra) = \pi_B(\neg Rb) = 0$ for all $a \in A$ and $b \in B$. This ensures

$$(h \circ \pi_A, a) \cong (h \circ \pi_A, a') \cong (h \circ \pi_B, b) \cong (h \circ \pi_B, b')$$

for all $a, a' \in A$ and $b, b' \in B$. By the isomorphism lemma, we can apply proposition 3.28 without partitioning A and B into smaller sets. We obtain that $\pi_A \equiv \pi_B$, independent of whether π_A and π_B share even a single valuation with regard to R.

Corollary 3.29. There are $\mathbb{N}^\infty$-interpretations π_A and π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$.

3.1.5 Polynomial Semirings

Finally, we analyze the applicability of the games G_m and G to interpretations which label the atomic facts with polynomials. More precisely, we examine the semirings PosBool$[X]$, $\mathbb{W}[X]$, $\mathbb{B}[X]$ as well as $\mathbb{S}[X]$ for finite sets X of variables and postpone the semiring $\mathbb{N}[X]$ to a detailed discussion within the subsequent section. Recall that only PosBool$[X]$ and $\mathbb{W}[X]$ admit infinitary operations. Hence, we only consider finite $\mathbb{B}[X]$- and $\mathbb{S}[X]$-interpretations and take into account the extended semirings $\mathbb{B}_\infty[\![X]\!]$ and $\mathbb{S}_\infty[X]$, which admit infinitary operations, if the finiteness of the interpretations affects the transferability of the games.

Crucial for the applicability of the game G_m as a method to prove m-equivalence is that PosBool$[X]$ is the only considered polynomial semiring which is fully idempotent. While the absence of multiplicative idempotence in both $\mathbb{B}[X]$ and $\mathbb{S}[X]$ readily follows from inequalities such as $x^2 \neq x$, it seems counterintuitive for the semiring $\mathbb{W}[X]$, which emerges from $\mathbb{B}[X]$ through collapsing exponents. As an example, we obtain $(x + y) \cdot (x + y) = x + xy + y \neq x + y$ in $\mathbb{W}[X]$, which proves that multiplicative idempotence is not fulfilled. Hence, m-equivalence is implied by Duplicator winning G_m for PosBool$[X]$-interpretations only, which also yields that elementary equivalence of PosBool$[X]$-interpretations follows from the game G. As observed for the semiring $\mathbb{N}^\infty$ in the previous section, full idempotence is not necessary for elementary equivalence to be implied by Duplicator winning the game G. Therefore, this implication remains to be analyzed for the other polynomial semirings.

In this context, $\mathbb{W}[X]$ is of particular interest, as the semiring differs in the applicability of the game G for proving elementary equivalence from the semirings discussed previously. Even though $\mathbb{W}[X]$ is not fully idempotent, $\mathbb{W}[X]$ is idempotent and fulfills a weakened form of multiplicative idempotence. As an example, we observe for the polynomial $x + y$ that $(x + y)^3 = x + xy + y = (x + y)^2$, which implies $(x + y)^2 = (x + y)^{2+n}$ for all $n \in \mathbb{N}$. This observation can be generalized to arbitrary polynomials in $\mathbb{W}[X]$ as follows.

Lemma 3.30. For any polynomial $p \in \mathbb{W}[X]$, it holds that $\prod_{i \in I} p = \prod_{j \in J} p$ for all index sets I, J with $|I| \geq |X|$ and $|J| \geq |X|$.

Proof. Due to distributivity and the definition of infinitary multiplication in $\mathbb{W}[X]$, we can write $\prod_{i\in I} p$ for each $p \in \mathbb{W}[X]$ and each set I as

$$\prod_{i\in I} p = \sum_{\substack{(m_i)_{i\in I}:\\ m_i\in p}} \prod_{i\in I} m_i.$$

Since a monomial in $\mathbb{W}[X]$ is a subset of X and multiplication of monomials corresponds to their union, we obtain for all I such that $|I| \geq X$

$$\prod_{i\in I} p = \sum_{\substack{p'\subseteq p:\\ |p'|\leq |I|}} \prod_{m\in p'} m = \sum_{\substack{p'\subseteq p:\\ |p'|\leq |X|}} \prod_{m\in p'} m,$$

which implies the claim. ☐

Recall that for fully idempotent semirings, it is possible to construct a winning strategy for Spoiler based on a separating formula, as unequal sums and products can always be attributed to unequal sets of summands or factors. Since $\mathbb{W}[X]$ is not multiplicatively idempotent, it does not hold for all families $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ of polynomials in $\mathbb{W}[X]$ that $\{p_i : i \in I\} = \{q_j : j \in J\}$ implies $\prod_{i\in I} p_i = \prod_{j\in J} q_j$. Hence, a universal quantifier in a separating formula cannot be eliminated within a single turn. However, lemma 3.30 enables the formulation of a weakened form of this implication, which ensures that Spoiler has a strategy to eliminate any universal quantifier in at most $|X|$ turns.

Corollary 3.31. Let $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ be families of polynomials in $\mathbb{W}[X]$ where X is a finite variable set. It must hold that $\prod_{i\in I} p_i = \prod_{j\in J} q_j$ if for all $p \in \mathbb{W}[X]$

$$|\{i \in I : p_i = p\}| = |\{j \in J : q_j = p\}| \text{ or}$$
$$|\{i \in I : p_i = p\}| \geq |X| \text{ and } |\{j \in J : q_j = p\}| \geq |X|.$$

Note that we made a similar observation for any semiring in which the infinitary operations emerge from the finite operations by considering the supremum or infimum. Due to the fact that different infinite cardinalities are not distinguished by this definition of the infinitary operations, in any such semiring sums and products must coincide if they share the same finite subfamilies of summands/factors. Although this allows Spoiler to eliminate each quantifier in a separating formula via finitely many turns, this property does not suffice in general for ensuring elementary equivalence if Duplicator wins G, as we proved for the semirings $\mathbb{V}$ and $\mathbb{L}$. But in the case of $\mathbb{W}[X]$, we can further bound the number of required turns by $|X|$. Intuitively, we derived that semirings with appropriate infinitary operations do not admit "counting beyond ω", whereas $\mathbb{W}[X]$, in particular, does not even enable "counting further than $|X|$".

Theorem 3.32. Let π_A and π_B be $\mathbb{W}[X]$-interpretations where X is a finite set of variables. Further, let $\bar{a} = (a_1, \dots, a_n) \in A^n$ and $\bar{b} = (b_1, \dots, b_n) \in B^n$. If $(\pi_A, \bar{a}) \not\equiv_m (\pi_B, \bar{b})$, then Spoiler wins the game $G_{|X| \cdot m}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Proof. The claim can be shown by induction on the structure of $\varphi(\bar{x})$. We only consider the case $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$ explicitly, as the remaining cases are analogous to the proof of theorem 3.5. Since $\varphi(\bar{x})$ separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$ by assumption, it holds that

$$\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!].$$

With 3.31, this implies that there is some $\ell \leq |X|$ and elements $a_{n+1}, \dots, a_{n+\ell} \in A$ (or $b_{n+1}, \dots, b_{n+\ell} \in B$) such that for all $b_{n+1}, \dots, b_{n+\ell} \in B$ (or $a_{n+1}, \dots, a_{n+\ell} \in A$, respectively) it holds that $\pi_A[\![\psi(\bar{a}, a_{n+i})]\!] \neq \pi_B[\![\psi(\bar{b}, b_{n+i})]\!]$ for some $1 \leq i \leq \ell$. In the game $G_{|X| \cdot m}(\pi_A, \bar{a}, \pi_B, \bar{b})$ where $m = \mathrm{qr}(\varphi(\bar{x}))$, Spoiler can pick such ℓ elements in any order. Independent of Duplicator's answers, the induction hypothesis can be applied to the formula $\psi(\bar{x}, x)$ and instantiations $(\bar{a}, a_{n+i})$ and $(\bar{b}, b_{n+i})$. Hence, Spoiler wins $G_{|X|(m-1)}(\pi_A, \bar{a}, a_{n+i}, \pi_B, \bar{b}, b_{n+i})$. Since $\ell \leq |X|$, Spoiler wins the remaining subgame in particular. $\square$

Corollary 3.33. If Duplicator wins the game $G(\pi_A, \pi_B)$ on $\mathbb{W}[X]$-interpretations, then it holds that $\pi_A \equiv \pi_B$.

Note that the number of turns Spoiler chooses in his winning strategy only depends on the number of variables and the quantifier rank of the separating formula. In particular, it does not depend on the $\mathbb{W}[X]$-interpretations the game is played on, unlike observed for the semiring $\mathbb{N}^\infty$.

If only finite interpretations are considered, it immediately follows from the isomorphism lemma that elementary equivalence is implied by the winning of Duplicator in the game G. Hence, we next examine whether this implication can be lifted to the semirings $\mathbb{B}_\infty[\![X]\!]$ and $\mathbb{S}_\infty[X]$, which extend $\mathbb{B}[X]$ and $\mathbb{S}[X]$ and admit infinitary operations. Since the polynomials in $\mathbb{B}_\infty[\![X]\!]$ and $\mathbb{S}_\infty[X]$ contain exponents, infinitely many polynomials can be generated by a single polynomial in the underlying monoids $(\mathbb{B}_\infty[\![X]\!], \cdot, 1)$ and $(\mathbb{S}_\infty[X], \cdot, 1)$. Hence, the proof for $\mathbb{W}[X]$ cannot be transferred to $\mathbb{B}_\infty[\![X]\!]$ and $\mathbb{S}_\infty[X]$. In fact, we can find a counterexample in $\mathbb{B}_\infty[\![X]\!]$.

Proposition 3.34. There are $\mathbb{B}_\infty[\![\{x\}]\!]$-interpretations π_A and π_B such that Duplicator wins $G(\pi_A, \pi_B)$ and $\pi_A \not\equiv_2 \pi_B$.

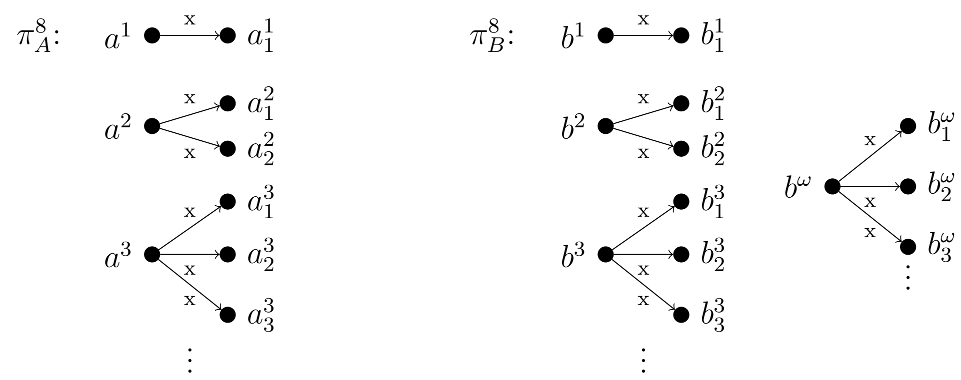
Proof. We construct $\mathbb{B}_\infty[\![\{x\}]\!]$-interpretations π_A^8 and π_B^8 over vocabulary $\tau = \{E\}$ where E is a binary relation symbol. The universes consist of elements

$$A = \{a^n \colon n \in \mathbb{N}_{>0}\} \cup \{a_i^n \colon n, i \in \mathbb{N}_{>0}, n \geq i\} \text{ and}$$
$$B = \{b^n \colon n \in \mathbb{N}_{>0}\} \cup \{b_i^n \colon n, i \in \mathbb{N}_{>0}, n \geq i\} \cup \{b^\omega\} \cup \{b_i^\omega \colon i \in \mathbb{N}_{>0}\}.$$

Further, the valuations of the $\{E\}$-literals over A and B are defined according to the following rules. Let $m, n, i \in \mathbb{N}_{>0}$ such that the following nodes are contained in A or B, respectively.

- $\pi_A^8(Ea^n a_i^n) = \pi_B^8(Eb^n b_i^n) = x$ and $\pi_B^8(Eb^\omega b_i^\omega) = x$
- $\pi_A^8(Ea^n a^m) = \pi_B^8(Eb^n b^m) = 1$ and $\pi_B^8(Eb^\omega b^n) = \pi_B^8(Eb^n b^\omega) = 1$
- $\pi_A^8(Ea^n a_i^m) = \pi_B^8(Eb^n b_i^m) = 1$ if $n \neq m$ and $\pi_B^8(Eb^\omega b_i^n) = \pi_B^8(Eb^n b_i^\omega) = 1$

The corresponding negated $\{E\}$-literals over A and B are valuated with 0. All remaining unnegated $\{E\}$-literals over A and B are valuated with 0 and their negations with 1.



It holds that $\pi_A^8 \not\equiv_2 \pi_B^8$, because

$$\pi_A[\![\exists x \forall y Exy]\!] = x + x^2 + x^3 + \ldots \neq x + x^2 + x^3 + \cdots + x^\infty = \pi_B[\![\exists x \forall y Exy]\!].$$

In order to show that Duplicator wins the game $G(\pi_A^8, \pi_B^8)$, we construct a partition $\mathcal{P}_A = \{A_1^n, A_2^n \colon n \in \mathbb{N}_{>0}\}$ of A and $\mathcal{P}_B = \{B_1^n, B_2^n \colon n \in \mathbb{N}_{>0} \cup \{\omega\}\}$ of B. Each A_1^n and B_1^n is a singleton consisting of the node a^n or b^n, respectively. Analogously, $B_1^\omega := \{b^\omega\}$, while the sets A_2^n and B_2^n contain all nodes a_i^n and b_i^n and B_2^ω consists of all b_i^ω. For each $m \in \mathbb{N}_{>0}$, we define a bijective mapping $g_m \colon \mathcal{P}_A \to \mathcal{P}_B$ by

$$g_m(A_i^n) := \begin{cases} B_i^n, & n < m \\ B_i^\omega, & n = m \\ B_i^{n-1}, & n > m \end{cases}.$$

Duplicator wins the game $G(\pi_A^8, \pi_B^8)$ as follows. Let m be the number of turns Spoiler chooses. Whenever Spoiler draws an element $a \in A_j^n$, she answers with some $b \in g_m(A_j^n)$. Analogously, each $b \in B_j^n$ is answered with some $a \in g_m^{-1}(B_j^n)$. Thereby, Duplicator only has to make sure that equalities with regard to the previously chosen elements are respected. Since $|A_j^n| = |g_m(A_j^n)|$, or both $|A_j^n| \geq m$ and $|g_m(A_j^n)| \geq m$ for each $A_j^n \in \mathcal{P}_A$, this can be realized for m turns and any resulting position must induce a local isomorphism. □

Hence, moving from polynomials to formal power series enables the construction of separable $\mathbb{B}_\infty[\![X]\!]$-interpretations on which Duplicator wins the game G, even if X only contains a single variable. It is noticeable that the counterexample also applies to $\mathbb{N}_\infty^\infty[\![X]\!]$. In the semiring $\mathbb{S}_\infty[X]$ however, the monomial x^1 would absorb all other monomials in the valuations of $\exists x \forall x Exy$, so it is not possible to construct an analogous counterexample for $\mathbb{S}_\infty[X]$. Similar to our observations with respect to $\mathbb{N}^\infty$, it can be shown that a winning strategy for Duplicator in G implies elementary equivalence of $\mathbb{S}_\infty[X]$-interpretations, while the number of required moves depends on the $\mathbb{S}_\infty[X]$-interpretations in question. Since $\mathbb{S}_\infty[X]$ is idempotent, Spoiler can eliminate any existential quantifier within a separating formula in a single turn. This does not hold for universal quantifiers, but we can make use of a similar invariant as in the proof for $\mathbb{N}^\infty$. Recall that the multiplication in $\mathbb{S}_\infty[X]$ is defined as

$$\prod_{i \in I} p_i = \sum_{\substack{(m_i)_{i \in I}: \\ m_i \in p_i}} \prod_{i \in I} m_i,$$

where $m_i \in p_i$ is supposed to indicate that the monomial m_i occurs in p_i. In particular, this implies that m_i is maximal with respect to the absorption order compared to the remaining monomials in p_i. Suppose that the valuation $\pi_A[\![\forall x \psi(x)]\!]$ for some formula $\psi(x)$ contains the monomial $x^n y^\infty$ where $n \in \mathbb{N}$. Then, for each $a \in A$ there must be a monomial m_a in $\pi_A[\![\psi(a)]\!]$ such that $x^n y^\infty = \prod_{a \in A} m_a$. From the exponent n of x we can conclude that there are at most n elements $a \in A$ such that m_a assigns x a non-zero exponent. Thus, Spoiler can eliminate the universal quantifier in at most $n + 1$ turns, similar to his winning strategy for $\mathbb{N}^\infty$-interpretations. This is why we incorporate an invariant in the induction which states that a monomial must be contained in the valuations of the separating formula whose exponents remain small. Moreover, we need to make sure that we fix a monomial with minimal exponents, analogous to the requirement that Spoiler draws elements in the $\mathbb{N}^\infty$-interpretation with the smaller valuation of the separating formula in the proof of theorem 3.24. In order to formalize this, we write $m \succcurlyeq_Y m'$ for monomials m, m' in $\mathbb{S}_\infty[X]$ and $Y \subseteq X$ if $m(y) \leq m'(y)$ for all $y \in Y$ where $m(y)$ and $m'(y)$ denote the exponent of the variable y in m and m'.

Theorem 3.35. Fix some $k \in \mathbb{N}_{>0}$. Let π_A and π_B be $\mathbb{S}^\infty[X]$-interpretations with elements $\bar{a} = (a_1, \ldots, a_n) \in A^n$ and $\bar{b} = (b_1, \ldots, b_n) \in B^n$. Further, let $\varphi(\bar{x})$ be a formula with $\pi_A[\![\varphi(\bar{a})]\!] =: p_A \neq p_B := \pi_B[\![\varphi(\bar{b})]\!]$ such that $p_A = 0$ or $p_B = 0$, or such that there is a monomial m contained in one of p_A and p_B and a non-empty set $Y \subseteq \{x \in X : m(x) \neq \infty\}$ satisfying

(1) $\sum_{y \in Y} m(y) < k$ and

(2) $m' \not\succcurlyeq_Y m$ for all monomials m' contained in the other polynomial.

Then, Spoiler has a winning strategy for the game $G_{k \cdot m}(\pi_A, \bar{a}, \pi_B, \bar{b})$ where m is the quantifier rank of $\varphi(\bar{x})$.

Proof. We show the claim by induction on the structure of $\varphi(\bar{x})$.

Case 1. If $\varphi(\bar{x})$ is a literal, it follows from the fact that $\varphi(\bar{x})$ is separating that the current position does not induce a local isomorphism. Thus, Spoiler wins the game $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. Suppose that $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) = m$. If either $\pi_A[\![\varphi(\bar{a})]\!]$ or $\pi_B[\![\varphi(\bar{b})]\!]$ is empty, say $\pi_A[\![\varphi(\bar{a})]\!] = 0$, then we can conclude that $\pi_A[\![\varphi_1(\bar{a})]\!] = \pi_A[\![\varphi_2(\bar{a})]\!] = 0$, whereas $\pi_B[\![\varphi_i(\bar{b})]\!] \neq 0$ for some $i \in \{1, 2\}$. As $\varphi_i(\bar{x})$ separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$ and is valuated with 0 by one of the interpretations, the induction hypothesis can be applied and we obtain that Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$. Otherwise, there must be a monomial m and a set Y of variables satisfying conditions (1) and (2). Assume w.l.o.g. that $m \in \pi_A[\![\varphi(\bar{a})]\!]$. Then, m must be contained in $\pi_A[\![\varphi_i(\bar{a})]\!]$ for some $i \in \{1, 2\}$. Any monomial m' in $\pi_B[\![\varphi_i(\bar{b})]\!]$ must either be part of $\pi_B[\![\varphi(\bar{b})]\!]$ or be absorbed by some monomial m'' in $\pi_B[\![\varphi(\bar{b})]\!]$. Since $m'' \not\succcurlyeq_Y m$ by assumption and $m'' \succcurlyeq m'$, both cases yield $m' \not\succcurlyeq_Y m$. In particular, this yields $m \notin \pi_B[\![\varphi_i(\bar{b})]\!]$, so $\varphi_i(\bar{x})$ separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. Hence, the induction hypothesis can be applied to $\varphi_i(\bar{x})$ with m and Y fulfilling (1) and (2) and we obtain that Spoiler wins the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 3. Suppose that $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) = m$. If either $\pi_A[\![\varphi(\bar{a})]\!]$ or $\pi_B[\![\varphi(\bar{b})]\!]$ is empty, say $\pi_A[\![\varphi(\bar{a})]\!] = 0$, then it must hold that $\pi_A[\![\varphi_i(\bar{a})]\!] = 0$ for some $i \in \{1, 2\}$, whereas $\pi_B[\![\varphi_i(\bar{b})]\!] \neq 0$. Hence, the induction hypothesis can be applied to $\varphi_i(\bar{x})$ and we obtain that Spoiler wins the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$. Otherwise, there must be a monomial m and a set Y of variables satisfying the conditions above. Assume w.l.o.g. that $m \in \pi_A[\![\varphi(\bar{a})]\!]$. Then, there is some $m_1 \in \pi_A[\![\varphi_1(\bar{a})]\!]$ and $m_2 \in \pi_A[\![\varphi_2(\bar{a})]\!]$ such that $m = m_1 \cdot m_2$. Since multiplication of monomials is defined by summation of their exponents, property (1) with regard to Y is clearly fulfilled by both m_1 and m_2. Suppose there were $m'_1 \in \pi_B[\![\varphi_1(\bar{b})]\!]$ and $m'_2 \in \pi_B[\![\varphi_2(\bar{b})]\!]$ with $m'_1 \succcurlyeq_Y m_1$ and $m'_2 \succcurlyeq_Y m_2$. Then, $m'_1 \cdot m'_2 \succcurlyeq_Y m_1 \cdot m_2$, so there would be some $m' \in \pi_B[\![\varphi(\bar{b})]\!]$ such that $m' \succcurlyeq_Y m$, which violates (2). Hence, for at least one $i \in \{1, 2\}$ it must hold that $m'_i \not\succcurlyeq_Y m_i$ for all $m'_i \in \pi_B[\![\varphi_i(\bar{b})]\!]$, which already implies that $\varphi_i(\bar{x})$ is separating. Hence, we can apply the induction hypothesis to $\varphi_i(\bar{x})$ and m_i with Y and infer that Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 4. Let $\varphi(\bar{x}) = \exists x\psi(\bar{x}, x)$, where $\text{qr}(\varphi(\bar{x})) = m$. In case one of $\pi_A[\![\varphi(\bar{a})]\!]$ and $\pi_B[\![\varphi(\bar{b})]\!]$) is equal to 0, suppose w.l.o.g. that $\pi_B[\![\varphi(\bar{b})]\!] \neq 0$. This implies $\pi_B[\![\psi(\bar{b}, b)]\!] \neq 0$ for some $b \in B$, Spoiler can pick in the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$. For all possible answers $a \in A$ it must hold that $\pi_A[\![\psi(\bar{a}, a)]\!] = 0$, which is why the induction hypothesis can be applied to $\psi(\bar{x}, x)$ and the updated position. We obtain that Spoiler wins $G_{k(m-1)}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$, so in particular, he wins the remaining subgame $G_{km-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$, since $km - 1 \geq k(m-1)$. If both $\pi_A[\![\varphi(\bar{a})]\!]$ and $\pi_B[\![\varphi(\bar{b})]\!]$ are non-zero, let m and Y be given as in the previous cases and assume again that $m \in \pi_A[\![\varphi(\bar{a})]\!]$. Then, there must be some $a \in A$ such that $m \in \pi_A[\![\psi(\bar{a}, a)]\!]$. Spoiler can choose this element $a \in A$ in the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$. If there were some $b \in B$ and $m' \in \pi_B[\![\psi(\bar{b}, b)]\!]$ with $m' \succcurlyeq_Y m$, then $m' \in \pi_B[\![\varphi(\bar{b})]\!]$ or $m'' \in \pi_B[\![\varphi(\bar{b})]\!]$ for some $m'' \succcurlyeq m'$ must hold. Both cases contradict condition (2), hence for any possible answer $b \in B$ the polynomials $\pi_A[\![\psi(\bar{a}, a)]\!]$ and $\pi_B[\![\psi(\bar{b}, b)]\!]$ have to fulfill condition (1) and (2) with respect to m and Y, which is why the induction hypothesis can be applied. Spoiler wins the game $G_{k(m-1)}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$ and, in particular, the remaining subgame.

Case 5. For $\varphi(\bar{x}) = \forall x\psi(\bar{x}, x)$ with $\text{qr}(\varphi(\bar{x})) = m$, the case $\pi_A[\![\varphi(\bar{a})]\!] = 0$ or $\pi_B[\![\varphi(\bar{b})]\!] = 0$ is analogous to the previous cases. Hence, let m and Y be witnesses for condition (1) and (2) and assume that $m \in \pi_A[\![\varphi(\bar{a})]\!]$. For any $a \in A$, there must be some $m_a \in \pi_A[\![\psi(\bar{a}, a)]\!]$ such that $m = \prod_{a \in A} m_a$. Since we assume that $\sum_{y \in Y} m(y) < k$, there are less than k elements $a_{n+1}, \ldots, a_{n+\ell} \in A$ such that $m_{a_{n+i}}(y) \neq 0$ for some $y \in Y$. Clearly, it holds that $\sum_{y \in Y} m_{a_{n+i}}(y) < k$ for all $y \in Y$ and $1 \leq i \leq \ell$. In the game $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$, Spoiler successively picks all these elements $a_{n+1}, \ldots, a_{n+\ell}$ in any order. Let $b_{n+1}, \ldots, b_{n+\ell}$ be Duplicator's answers. If there is some $1 \leq i \leq \ell$ such that for all $m_{b_{n+i}} \in \pi_B[\![\psi(\bar{b}, b_{n+i})]\!]$ it holds that $m_{b_{n+i}} \not\succcurlyeq_Y m_{a_{n+i}}$, then the induction hypothesis can be applied to $\psi(\bar{x}, x)$ and the tuples $(\bar{a}, a_{n+i})$ and $(\bar{b}, b_{n+i})$. We can infer that Spoiler wins the game $G_{k(m-1)}(\pi_A, \bar{a}, a_{n+i}, \pi_B, \bar{b}, b_{n+i})$ and thus also the remaining subgame, as $\ell < k$. Otherwise, for each $1 \leq i \leq \ell$ there is some $m_{b_{n+i}} \in \pi_B[\![\psi(\bar{b}, b_{n+i})]\!]$ such that $m_{b_{n+i}} \succcurlyeq_Y m_{a_{n+i}}$. Towards a contradiction, suppose that for each $b \in B \setminus \{b_{n+1}, \ldots, b_{n+\ell}\}$ there is some $m_b \in \pi_B[\![\psi(\bar{b}, b)]\!]$ such that $m_b(y) = 0$ for each $y \in Y$. Then, the monomial $m' = \prod_{1 \leq i \leq n} m_{b_{n+i}} \cdot \prod_{b \in B \setminus \{b_{n+1}, \ldots, b_{n+\ell}\}} m_b$ would be contained in $\pi_B[\![\varphi(\bar{b})]\!]$ or absorbed by some monomial included in $\pi_B[\![\varphi(\bar{b})]\!]$. Both cases violate (2), because $m' \succcurlyeq_Y m$. Hence, Spoiler can pick some $b_{n+\ell+1}$ such that for all $m_{b_{n+\ell+1}} \in \pi_B[\![\psi(\bar{b}, b_{n+\ell+1})]\!]$ there is some $y \in Y$ with $m_{b_{n+\ell+1}}(y) > 0$.

But for all answers $a_{n+\ell+1} \in A \setminus \{a_{n+1}, \dots, a_{n+\ell}\}$ we have that $m_{a_{n+\ell+1}}(y) = 0$ for all $y \in Y$. This is why we can apply the induction hypothesis to $\psi(\bar{x}, x)$ with instantiations $(\bar{a}, a_{n+\ell+1})$ and $(\bar{b}, b_{n+\ell+1})$, as $m_{n+\ell+1}$ fulfills (1) and (2) with regard to Y, and conclude that Spoiler wins the game $G_{k(m-1)}(\pi_A, \bar{a}, a_{n+\ell+1}, \pi_B, \bar{b}, b_{n+\ell+1})$, so the remaining subgame as well, since $k(m-1) \leq km - \ell - 1$. □

It remains to show that the invariant we used to construct the winning strategy for Spoiler is fulfilled by any sentence φ separating $\mathbb{S}^{\infty}[X]$-interpretations π_A and π_B.

Lemma 3.36. For any two non-empty distinct polynomials $p, q \in \mathbb{S}^{\infty}[X]$, there is a monomial m contained in p or q and a non-empty set $Y \subseteq \{x \in X : m(x) \neq \infty\}$ such that $m' \not\succeq_Y m$ for all monomials m' the other polynomial consists of.

Proof. Since $p \neq q$ by assumption, there must be a monomial m which is contained in one of p and q such that $m \not\succeq m'$ for all monomials m' in the other polynomial. Both p and q are non-empty, hence m must assign an exponent $n \neq \infty$ to some variable, as the monomial which assigns ∞ to each variable is absorbed by any monomial. Hence, we can choose Y as the set of variables m assigns an exponent other than ∞. In this case, $m' \not\succeq m$ implies $m' \not\succeq_Y m$ for all m' contained in the polynomial m is not part of. □

Corollary 3.37. Given $\mathbb{S}_{\infty}[X]$-interpretations π_A and π_B, it holds that $\pi_A \equiv \pi_B$ if Duplicator wins the game $G(\pi_A, \pi_B)$.

Summing up, we derived that PosBool$[X]$ is the only considered polynomial semiring for which Duplicator winning G_m implies m-equivalence, while elementary equivalence follows from the game G for PosBool$[X]$, $\mathbb{W}[X]$ and $\mathbb{S}_{\infty}[X]$.

As for inferring separability of polynomial interpretations using G or G_m, we refer to the counterexamples provided in [GM21] which illustrate that elementary equivalence does not imply isomorphism of finite interpretations in these semirings. We obtain that neither G_m nor G can be used to show the existence of a separating sentence for any of the considered polynomial semirings.

Proposition 3.38. For any $\mathcal{K} \in \{\text{PosBool}[X], \mathbb{W}[X], \mathbb{B}[X], \mathbb{S}[X]\}$ there are finite $\mathcal{K}$-interpretations π_A and π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$.

3.2 Bijection and Counting Game

A major observation we derived in the analysis of the standard Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations is that for $k \in K$, the number of elements $a \in A$

such that $\pi_A[\![\psi(a)]\!] = k$ can affect both $\pi_A[\![\exists x\psi(x)]\!]$ and $\pi_A[\![\forall x\psi(x)]\!]$. In this manner, a single quantifier can reflect the number of elements that lead to a certain semiring valuation, whereas in classical FO, counting can only be accomplished by nesting quantifiers. This observation gives rise to examining the transferability of variants of the Ehrenfeucht-Fraïssé game capturing logics which include a counting mechanism. A well-known variant, the *k-pebble bijection game*, goes back to Hella [Hel92] and provides a game-theoretic characterization of elementary equivalence in C^k, the k-variable fragment of FO with counting quantifiers, on finite structures. When applying the variant to $\mathcal{K}$-interpretations, we slightly modify the game rules and do not make use of the notion of pebbles. Instead, we fix the number of moves at the beginning, as we classify the first-order formulae by their quantifier rank instead of their width.

Definition 3.39. In the game $BG_m(\pi_A, \pi_B)$ on $\mathcal{K}$-interpretations π_A and π_B, each play comprises m turns. After the i-th round, some position $(a_1, \dots, a_i, b_1, \dots, b_i)$ is reached and the play proceeds according to $BG_{m-i}(\pi_A, a_1, \dots, a_i, \pi_B, b_1, \dots, b_i)$ as follows: Duplicator has to provide a bijection $h\colon A \to B$. If such a bijection does not exist, i.e. $|A| \neq |B|$, Spoiler wins immediately. Otherwise, Spoiler chooses some $a \in A$ and the pair $(a, h(a))$ is added to the current position. Duplicator wins the play if, and only if, the resulting position $(a_1, \dots, a_m, b_1, \dots, b_m)$ induces a local isomorphism between π_A and π_B.

Soundness of BG_m. Recall the $\mathbb{N}$-interpretations π_A^1 and π_B^1, on which Spoiler loses G_1 despite separability by $\exists xRx$. Spoiler has a winning strategy for the game BG_1, as for any bijection $h\colon A \to B$, it must hold that $\{h(a_1), h(a_2)\} \cap \{b_2, b_3\} \neq \emptyset$. Hence, Spoiler is able to pick some $a \in A$ such that $\pi_A^1(Ra) = 1 \neq 2 = \pi_B^1(Rh(a))$, which causes him to win $BG_1(\pi_A^1, \pi_B^1)$.

$$\pi_A^1 : \begin{array}{c||c|c} A & R & \neg R \\ \hline\hline a_1 & 1 & 0 \\ a_2 & 1 & 0 \\ a_3 & 2 & 0 \end{array} \qquad \begin{array}{c} h\colon \\ \to \\ \not\to \\ \to \end{array} \qquad \pi_B^1 : \begin{array}{c||c|c} B & R & \neg R \\ \hline\hline b_1 & 1 & 0 \\ b_2 & 2 & 0 \\ b_3 & 2 & 0 \end{array}$$

In this way, the game BG_m reflects that the precise number of elements leading to a certain semiring valuation may ensure the separability by a first-order sentence in semiring semantics. This allows us to drop the requirement of full idempotence, which is necessary for the soundness of G_m, and to infer m-equivalence if Duplicator has a winning strategy for the game BG_m for arbitrary semirings $\mathcal{K}$.

Theorem 3.40. Given any semiring $\mathcal{K}$ and $\mathcal{K}$-interpretations π_A, π_B with elements $\bar{a} \in A^n$ and $\bar{b} \in B^n$, it must hold that $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$ if Duplicator wins the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Proof. Let $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ be a formula of quantifier rank m with $\bar{x} = (x_1, \dots, x_n)$ such that $\pi_A[\![\varphi(\bar{a})]\!] \neq \pi_B[\![\varphi(\bar{b})]\!]$ for some $\bar{a} \in A^n$ and $\bar{b} \in B^n$. We show by induction on the structure of $\varphi(\bar{x})$ that Spoiler wins the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 1. If $\varphi(\bar{x})$ is a literal, the current position cannot induce a local isomorphism. Consequently, Spoiler wins the game $BG_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. For $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$ with $\circ \in \{\vee, \wedge\}$ and $\mathrm{qr}(\varphi(\bar{x})) = m$, we can infer that one of $\psi(\bar{x})$ and $\vartheta(\bar{x})$ must already separate $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. Thus, it follows immediately from the induction hypothesis that Spoiler wins $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 3. Let $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ and $\mathrm{qr}(\varphi(\bar{x})) = m$. Towards a contradiction, suppose there was a bijection $h\colon A \to B$ such that

$$\pi_A[\![\psi(\bar{a}, a)]\!] = \pi_B[\![\psi(\bar{b}, h(a))]\!]$$

for all $a \in A$. With invariance under bijections of (infinitary) addition and multiplication in $\mathcal{K}$, this would immediately imply

$$\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \sum_{a \in A} \pi_B[\![\psi(\bar{b}, h(a))]\!] = \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \text{ and}$$
$$\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \prod_{a \in A} \pi_B[\![\psi(\bar{b}, h(a))]\!] = \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!],$$

contradicting the assumption that $\varphi(\bar{x})$ separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. So for any bijection h Duplicator provides in $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, Spoiler can pick some $a \in A$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq \pi_B[\![\psi(\bar{b}, h(a))]\!]$. Thus, the induction hypothesis can be applied to $\psi(\bar{x}, x)$ and the updated position $(\bar{a}, a, \bar{b}, h(a))$ and we can infer that Spoiler wins the remaining subgame $BG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, h(a))$. ☐

Completeness of BG_m. In section 3.1, we derived that in none of the examined semirings, a winning strategy for Spoiler in G_m suffices in general to deduce m-separability. Since moving from G_m to BG_m facilitates Spoiler to win, the counterexamples can be transferred and we can conclude that BG_m is not appropriate as a proof method for m-separability, either.

Moreover, when applying BG_m instead of G_m to $\mathcal{K}$-interpretations, we are still encountering issues related to counting but they are shifted to the converse implication, i.e., they affect the completeness instead of the soundness of the game. The game BG_m ensures that Spoiler wins the game played on $(\pi_A, \bar{a}, \pi_B, \bar{b})$ in case there exists some $k \in K$ and a formula $\psi(\bar{x}, x)$ of quantifier rank $m-1$ such that a different number of elements in A and B lead to a valuation of $\psi(\bar{x}, x)$ with k in $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$, respectively. Thus, the game presumes m-separability whenever such k and $\psi(\bar{x}, x)$ exist, which is not appropriate for a multitude of semirings, in particular if full idempotence is satisfied. As an example, regard the valuations in the $\mathbb{N}$-interpretations π_A^1 and π_B^1 as elements from the min-max semiring $\mathcal{K}_2 := (\{0, 1, 2\}, \max, \min, 0, 2)$ induced by the usual order on $\{0, 1, 2\}$. Due to full idempotence, we can apply the game G_1 and infer 1-equivalence. However, Duplicator does not win the game BG_1.

Proposition 3.41. For any fully idempotent $\mathcal{K}$ and any $m \in \mathbb{N}_{>0}$, there are m-equivalent $\mathcal{K}$-interpretations π_A and π_B such that Spoiler wins $BG_1(\pi_A, \pi_B)$.

Proof. Let π_A^m and π_B^{m+1} be $\mathcal{K}$-interpretations over the empty vocabulary with universes A and B such that $|A| = m$ and $|B| = m + 1$. Clearly, Duplicator wins the game $G_m(\pi_A^m, \pi_B^{m+1})$. Since $\mathcal{K}$ is fully idempotent by assumption, this implies $\pi_A \equiv_m \pi_B$. However, Spoiler wins the game $BG_1(\pi_A, \pi_B)$, as there is no bijection from A to B. $\square$

An observation to be noted is that whether the existence of k and $\psi(\bar{x}, x)$ with $|\{a \in A \colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}| \neq |\{b \in B \colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}|$ ensures m-separability of $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$ does not only depend on the semiring but also on the element k as well as on the precise number of elements leading to a valuation of $\psi(\bar{x}, x)$ with k in $(\pi_A, \bar{a})$ and $(\pi_B, \bar{b})$. In the Viterbi semiring, for instance, multiple occurrences of the valuations 0 and 1 might not lead to separation compared to any other valuation in between.

$\pi_A^9 :$

A	R	$\neg R$
a_1	1	0
a_2	1	0
a_3	0.5	0

$\equiv_1$ $\pi_B^9 :$

B	R	$\neg R$
b_1	1	0
b_2	0.5	0

$\not\equiv_1$ $\pi_C^9 :$

C	R	$\neg R$
c_1	1	0
c_2	0.5	0
c_3	0.5	0

While $\pi_A^9 \equiv_1 \pi_B^9$ follows from the sufficient criterion for 1-equivalence of $\mathbb{V}$-interpretations relying on a single unary relation symbol, which we derived in proposition 3.13, π_B^9 and π_C^9 are separable by $\forall x Rx$. For any element k in the Viterbi semiring, we either obtain $k^2 = k$ or $k^n \neq k$ for any $k \in \mathbb{V}$, i.e., each element either behaves like in a fully idempotent semiring, or it never generates the same element via distinct exponent. By contrast, we observed, for example, for the polynomial semiring $\mathbb{W}[X]$ that $p^{|X|+n} = p^{|X|}$ is true for each $n \in \mathbb{N}$, although multiplicative idempotence is not fulfilled. As we showed that a winning strategy for Duplicator in $G_{|X| \cdot m}$ implies m-equivalence, we can observe in the subsequent example, where $X = \{x, y\}$, that the precise number of occurrences of semiring valuations might be crucial for m-equivalence.

$$\pi_A^{10} : \begin{array}{c||c|c} A & R & \neg R \\ \hline a_1 & x+y & 0 \end{array} \not\equiv_1 \pi_B^{10} : \begin{array}{c||c|c} B & R & \neg R \\ \hline\hline b_1 & x+y & 0 \\ \hline b_2 & x+y & 0 \end{array} \equiv_1 \pi_C^{10} : \begin{array}{c||c|c} C & R & \neg R \\ \hline\hline c_1 & x+y & 0 \\ \hline c_2 & x+y & 0 \\ \hline c_3 & x+y & 0 \end{array}$$

Intuitively, the classical Ehrenfeucht-Fraïssé game relies on the assumption that counting cannot be realized at all in the logic to be captured. On the contrary, we assume when applying the bijection game to $\mathcal{K}$-interpretations that counting semiring valuations is possible to the full extent. However, neither of the assumptions is justified for first-order logic with semiring semantics, as different semirings might exhibit more complex behaviors in between, which constitutes a main difficulty in search of a game-theoretic characterization. For instance, only certain semiring elements might admit counting, or counting might only be realizable up to a certain bound, as illustrated by the previous examples.

In order to partially account for this observation, the winning condition of the bijection game can be extenuated by incorporating a bound up to which the considered semiring admits counting. We formalize the existence of such a bound by introducing the notion of *κ-idempotence* for cardinal numbers κ, which can be considered a weakened form of full idempotence.

Definition 3.42. A semiring $\mathcal{K}$ is said to be *κ-idempotent* where $\kappa \in \mathrm{Cn}$ is a cardinal number if for each $k \in \mathcal{K}$ it holds that

$$\sum_{i\in I} k = \sum_{j \in J} k \text{ and } \prod_{i \in I} k = \prod_{j \in J} k$$

for all index sets I, J such that $|I| \geq \kappa$ and $|J| \geq \kappa$.

As an example, each semiring with infinitary operations emerging from the finite operations by applying supremum or infimum is ω-idempotent. Further, we have shown in section 3.1.5 that $\mathbb{W}[X]$ is $|X|$-idempotent in order to prove that a winning strategy for Duplicator in $G_{|X|\cdot m}$ implies m-equivalence and thus that elementary equivalence is ensured if Duplicator wins G. Yet, we can derive a more general result relying on κ-idempotence, so a game which Duplicator wins more often than BG_m or $G_{\kappa m}$ but which still ensures m-equivalence. To this end, we adjust the *counting game* introduced in [IL90] by Immerman and Lander, which also captures the logic C^k on finite classical structures.

Definition 3.43. Let $\kappa \in \mathrm{Cn}$ be a cardinal number. After the i-th of m turns in the game $CG_m^\kappa(\pi_A, \pi_B)$, the play is at position $(a_1, \ldots, a_i, b_1, \ldots, b_i)$. In the remaining subgame, which is denoted as $CG_{m-i}^\kappa(\pi_A, a_1, \ldots, a_i, \pi_B, b_1, \ldots, b_i)$, Spoiler chooses a set $X \subseteq A$ or $X \subseteq B$ such that $|X| \leq \kappa$ and Duplicator has to react with a subset Y of the other universe such that $|X| = |Y|$. Afterwards, Spoiler picks some $y \in Y$, Duplicator must answer with some element $x \in X$ and the pair (x, y) is added to the current position. The play is won by Duplicator if, and only if, the resulting position $(a_1, \dots, a_m, b_1, \dots, b_m)$ induces a local isomorphism between π_A and π_B.

Note that the game CG^1_m coincides with the classical Ehrenfeucht-Fraïssé game G_m. As a generalization of our previous result stating that a winning strategy for Duplicator in G_m ensures m-equivalence for fully idempotent, i.e., 1-idempotent semirings, m-equivalence in κ-idempotent semirings follows from CG^κ_m.

Theorem 3.44. Let $\mathcal{K}$ be a κ-idempotent semiring and π_A, π_B be $\mathcal{K}$-interpretations. If Duplicator has a winning strategy for $CG^\kappa_m(\pi_A, \pi_B)$, then it must hold that $\pi_A \equiv_m \pi_B$.

Proof. We prove by induction on $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ that $\pi_A[\![\varphi(\bar{a})]\!] \neq \pi_B[\![\varphi(\bar{b})]\!]$ implies that Spoiler wins the game $CG^\kappa_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ where $m = \mathrm{qr}(\varphi(\bar{x}))$ for each $\bar{a} \in A^n$ and $\bar{b} \in B^n$. We only consider the case $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$, as the remaining cases coincide with those in 3.40. As $\varphi(\bar{x})$ is separating, it holds that

$$\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \text{ or}$$
$$\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!].$$

Due to invariance under partitions and κ-idempotence, there must be some $k \in K$ such that

$$|\underbrace{\{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}}_{=:A^k_{\psi,\bar{a}}}| \neq |\underbrace{\{b \in B\colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}}_{=:B^k_{\psi,\bar{b}}}|,$$

where $|A^k_{\psi,\bar{a}}| < \kappa$ or $|B^k_{\psi,\bar{b}}| < \kappa$. Assume w.l.o.g. that $|A^k_{\psi,\bar{a}}| < |B^k_{\psi,\bar{b}}|$. Spoiler has the following winning strategy for the game $CG^\kappa_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. First, he chooses some $B' \subseteq B^k_{\psi,\bar{b}}$ with $|B'| = |A^k_{\psi,\bar{a}}| + 1 \leq \kappa$. For any possible answer $A' \subseteq A$ of Duplicator, there must be some $a \in A'$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq k$, since $|A'| = |B'| > |A^k_{\psi,\bar{a}}|$. Spoiler picks this element a. Independent of Duplicator's response $b \in B^k_{\psi,\bar{b}}$, it holds that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq k = \pi_B[\![\psi(\bar{b}, b)]\!]$. Hence, the $\mathcal{K}$-interpretations with updated position can be separated by $\psi(\bar{x}, x)$ and the induction hypothesis can be applied. Consequently, Spoiler wins the remaining subgame $CG^\kappa_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$. $\square$

It is noticeable that in the constructed winning strategy for CG_m^ω Spoiler only chooses finite sets, so m-equivalence of ω-idempotent semirings is already ensured if Duplicator has a strategy to win each play in CG_m^ω where Spoiler only chooses finite sets.

The games CG_m^κ can indeed be considered an extenuated version of BG_m, as the m-turn bijection game is equivalent to the m-turn counting game CG_m without restrictions on the cardinalities of the sets to be chosen, like the original versions of the k-pebble bijection and counting game on classical structures.

Theorem 3.45. Let π_A and π_B be $\mathcal{K}$-interpretations. From any position $(\bar{a}, \bar{b})$, Duplicator wins the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ if, and only if, she wins $CG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Proof. By induction on $m \in \mathbb{N}$, we prove the claim for all positions given by $\bar{a} \in A^n$ and $\bar{b} \in B^n$. For $m = 0$ the claim is clearly true, as the games BG_m and CG_m use the same winning condition.

If Duplicator has a winning strategy for $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, there is a bijection $h\colon A \to B$ such that Duplicator wins $BG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, h(a))$ for all $a \in A$. Based on h, Duplicator can win $CG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ as follows. If Spoiler chooses some set $X \subseteq A$, she answers with $h(X)$ and $h^{-1}(y)$ where $y \in h(X)$ is the element Spoiler picks afterwards. Otherwise, $X \subseteq B$ and Duplicator can react with $h^{-1}(X)$ and pick $h(y)$ after Spoiler chooses $y \in h^{-1}(X)$. In both cases, the subsequent position is given by $(\bar{a}, a, \bar{b}, h(a))$ for some $a \in A$. As Duplicator has a winning strategy for the game $BG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, h(a))$, she also wins the remaining subgame $CG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, h(a))$ by induction hypothesis.

Assume now that Duplicator wins $CG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. Let $\sim$ be the smallest relation on $A \dot\cup B$ such that $x \sim y$ if Duplicator wins $CG_{m-1}(\pi_X, \bar{x}, x, \pi_Y, \bar{y}, y)$, where $(\pi_X, \bar{x}), (\pi_Y, \bar{y}) \in \{(\pi_A, \bar{a}), (\pi_B, \bar{b})\}$. It can be easily verified that $\sim$ is an equivalence relation. Towards a contradiction, suppose there was an equivalence class Z such that $|Z \cap A| \neq |Z \cap B|$, w.l.o.g. let $|Z \cap A| < |Z \cap B|$. Then, Spoiler would win $CG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by choosing $Z \cap B$. For any possible response $A' \subseteq A$ of Duplicator, there is some $a \in A'$ with $a \notin Z \cap A$, which Spoiler can pick afterwards. Hence, for any equivalence class Z with respect to $\sim$, it holds that $|Z \cap A| = |Z \cap B|$. In the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, Duplicator can provide the combination of arbitrary bijections $h_Z\colon Z \cap A \to Z \cap B$. For any updated position $(\bar{a}, a, \bar{b}, b)$, it must hold that $a \sim b$, hence Duplicator wins the game $CG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$ and, by induction hypothesis, $BG_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$ as well. $\square$

As opposed to the previous semirings, for which we observed that first-order logic with semiring semantics is not expressive enough to separate any two $\mathcal{K}$-interpretations on which Spoiler wins BG_m, the bijection game constitutes a characterization of m-equivalence with regard to the natural semiring $\mathbb{N}$ and the polynomial semiring $\mathbb{N}[X]$, as we will prove in the following.

3.2.1 The Semirings $\mathbb{N}$ and $\mathbb{N}[X]$

The natural semiring $(\mathbb{N}, +, \cdot, 0, 1)$ does not admit infinitary operations, which is why we restrict the following analysis to finite interpretations. In order to show that m-equivalence of finite $\mathbb{N}$-interpretations is captured by the game BG_m or, equivalently, CG_m, we aim to construct characteristic sentences which ensure the winning of Duplicator in BG_m. To this end, we apply a combinatorial lemma, which was proven in [GM21] for the purpose of constructing characteristic sentences capturing elementary equivalence of $\mathbb{N}$-interpretations. Instead of applying the lemma to prove the existence of an isomorphism between two given $\mathbb{N}$-interpretations, we will use it to derive bijections which ensure the winning of Duplicator in BG_m.

Lemma 3.46. For each $\ell, d \in \mathbb{N}$, there is some $e \in \mathbb{N}$ such that for all tuples $(r_1, \dots r_{\ell'}), (s_1, \dots, s_{\ell'}) \in \mathbb{N}^{\ell'}$ with $\ell' < \ell$ and $r_i, s_i < d$ for $1 \leq i \leq \ell'$

$$\sum_{i=1}^{\ell'} r_i^e = \sum_{i=1}^{\ell'} s_i^e$$

implies that there is a permutation $\sigma \in S_{\ell'}$ such that $r_i = s_{\sigma(i)}$ for all $1 \leq i \leq \ell'$.

The lemma is central for the construction of characteristic sentences, as it allows us to infer the equality of the individual summands based on the equality of two sums. Recall that one of the main reasons why we cannot construct a separating sentence based on a winning strategy for Spoiler in general is that even though there is some element $a \in A$ resulting in the valuation $\pi_A[\![\varphi(a)]\!]$ with regard to some formula $\varphi(x)$ which cannot be duplicated by any $b \in B$, this might not be reflected by the valuations of $\exists x\varphi(x)$ and $\forall x\varphi(x)$ in π_A and π_B. Applying the lemma above allows us to conclude that a formula $\varphi(x)$ leads to exactly the same valuations in π_A and π_B for different instantiations of x, if $|A| = |B|$ and π_A and π_B agree on the sentence

$$\exists x\varphi(x)^e = \exists x \underbrace{\varphi(x) \land \cdots \land \varphi(x)}_{e \text{ times}},$$

where e is chosen large enough, depending on π_A, π_B and $\varphi(x)$.

Crucial for the application of the lemma is that the length of the tuples as well as the entries themselves are bounded. Hence, we associate with each $m, n \in \mathbb{N}$

multiple characteristic formulae $\chi_{\bar{c}}^m(\bar{x})$ of quantifier rank m with $\bar{x} = (x_1, \ldots, x_n)$ which depend on a pair of constants $\bar{c} = (c_1, c_2) \in \mathbb{N}^2$ and apply only to certain $\mathbb{N}$-interpretations which are bounded by $\bar{c}$. More precisely, we will show that $\pi_A[\![\chi_{\bar{c}}^m(\bar{x})]\!] = \pi_B[\![\chi_{\bar{c}}^m(\bar{b})]\!]$ implies that Duplicator wins $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ if π_A and π_B only include valuations smaller than c_1 and if their universes are of cardinality less than c_2. Each $\chi_{\bar{c}}^m(\bar{x})$ contains the subformula $\vartheta_{\bar{c}}^m(\bar{x})$, which is supposed to ensure the winning of Duplicator in BG_m in the case $|A| = |B|$. Accordingly, $\vartheta_{\bar{c}}^0(\bar{x})$ shall characterize the winning condition of the bijection game. To implement this, we fix an enumeration $L_1(\bar{x}), \ldots, L_k(\bar{x})$ of the τ-literals in $\mathrm{Lit}_n(\tau)$ and make use of the idea from [GM21], that is, we represent the valuations of the τ-literals as digits in a number system. Choosing the radix large enough ensures that the single valuations coincide in π_A and π_B.

$$\vartheta_{\bar{c}}^0(\bar{x}) := \bigvee_{1 \le i \le k} \underbrace{(L_i(\bar{x}) \vee \cdots \vee L_i(\bar{x}))}_{c_1^{i-1} \text{ times}}$$

Based on $\vartheta_{\bar{c}}^{m-1}(\bar{x}, x)$, we define $\vartheta_{\bar{c}}^m(\bar{x})$ such that $\pi_A[\![\vartheta_{\bar{c}}^m(\bar{a})]\!] = \pi_B[\![\vartheta_{\bar{c}}^m(\bar{b})]\!]$ ensures that $(\pi_A[\![\vartheta_{\bar{c}}^{m-1}(\bar{a}, a)]\!])_{a \in A \setminus \{a_1, \ldots, a_n\}}$ and $(\pi_B[\![\vartheta_{\bar{c}}^{m-1}(\bar{b}, b)]\!])_{b \in B \setminus \{b_1, \ldots, b_n\}}$ only differ by some permutation according to lemma 3.46. Let

$$\vartheta_{\bar{c}}^m(\bar{x}) := \exists x\big(\big(\bigwedge_{1 \le i \le n} x \neq x_i \wedge \vartheta_{\bar{c}}^{m-1}(\bar{x}, x)\big)^{e_{m-1}}\big),$$

where e_{m-1} is chosen according to lemma 3.46 with respect to $\ell := \max(c_2, 4)$ and d_{m-1} which is inductively defined by $d_0 := c_1^{k+1}$ and $d_{i+1} := c_2 \cdot d_i^{e_i}$ for $i > 0$. Note that this definition ensures that $d_m > \pi_A[\![\vartheta_{\bar{c}}^m(\bar{a})]\!]$ and $d_m > \pi_B[\![\vartheta_{\bar{c}}^m(\bar{b})]\!]$ for all $m \in \mathbb{N}$.

In order to drop the assumption $|A| = |B|$ the formulae $\vartheta_{\bar{c}}^m(\bar{x})$ rely on, since lemma 3.46 presumes tuples of the same length, we additionally encode in $\chi_{\bar{c}}^m(\bar{x})$ that the universes must be of the same cardinality. Having defined the sequence $(e_m)_{m \in \mathbb{N}}$ of exponents with respect to tuples of length smaller than $\max(c_2, 4)$ allows us to reuse them for this purpose.

$$\begin{aligned}
\chi_{\bar{c}}^0(\bar{x}) &:= \vartheta_{\bar{c}}^0(\bar{x}) \\
\chi_{\bar{c}}^m(\bar{x}) &:= (\exists x(x = x) \vee \exists x(x = x) \vee \vartheta_{\bar{c}}^m(\bar{x}))^{e_m}
\end{aligned}$$

Theorem 3.47. Let $c_1, c_2, m \in \mathbb{N}$. For all (finite) $\mathbb{N}$-interpretations π_A, π_B and $\bar{a} \in A^n$, $\bar{b} \in B^n$ such that $\max \mathrm{img}(\pi_A) \cup \mathrm{img}(\pi_B) < c_1$ and $|A| < c_2$, $|B| < c_2$, the following are equivalent:

(1) Duplicator wins $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$

(2) $\pi_A[\![\chi^m_{\bar{c}}(\bar{a})]\!] = \pi_B[\![\chi^m_{\bar{c}}(\bar{b})]\!]$

(3) $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$

Proof. By theorem 3.40, it holds that (1) $\Rightarrow$ (3). Since $\mathrm{qr}(\chi^m_{\bar{c}}(\bar{x})) = m$ by definition, we can immediately infer implication (3) $\Rightarrow$ (2). It remains to show (2) $\Rightarrow$ (1), which we prove by induction on m for all tuples $\bar{a} \in A^n$ and $\bar{b} \in B^n$ simultaneously. To this end, let π_A, π_B and $\bar{c} = (c_1, c_2)$ be given as above.

Case 1 ($m = 0$). As $\pi_A(L_i(\bar{a})) < c_1$ and $\pi_B(L_i(\bar{b})) < c_1$ for all $1 \leq i \leq k$ by assumption, $\pi_A[\![\chi^0_{\bar{c}}(\bar{a})]\!] = \pi_B[\![\chi^0_{\bar{c}}(\bar{b})]\!]$ implies that $\pi_A(L_i(\bar{a})) = \pi_B(L_i(\bar{b}))$ for all $1 \leq i \leq k$. Thus, the tuples $\bar{a}$ and $\bar{b}$ induce a local isomorphism between π_A and π_B, which is why Duplicator wins the game $BG_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2 ($m > 0$). By definition of d_m, it holds that $d_m \geq c_2 > \max(|A|, |B|) = \max(\pi_A[\![\exists x(x = x)]\!], \pi_B[\![\exists x(x = x)]\!])$. Further, we constructed d_m such that $d_m > \max(\pi_A[\![\vartheta^m_{\bar{c}}(\bar{a})]\!], \pi_B[\![\vartheta^m_{\bar{c}}(\bar{b})]\!])$. Since e_m has been chosen with respect to $\ell = \max(c_2, 4) > 3$ and d_m, we can apply lemma 3.46 and conclude that the triples

$$(\pi_A[\![\exists x(x = x)]\!], \pi_A[\![\exists x(x = x)]\!], \pi_A[\![\vartheta^m_{\bar{c}}(\bar{a})]\!]) \text{ and}$$
$$(\pi_B[\![\exists x(x = x)]\!], \pi_B[\![\exists x(x = x)]\!], \pi_B[\![\vartheta^m_{\bar{c}}(\bar{b})]\!])$$

only differ by some permutation σ. Suppose that σ is not the identity mapping. Then we would have that $\pi_A[\![\exists x(x = x)]\!] = \pi_B[\![\exists x(x = x)]\!] = \pi_B[\![\vartheta^m_{\bar{c}}(\bar{b})]\!]$ and $\pi_A[\![\vartheta^m_{\bar{c}}(\bar{a})]\!] = \pi_B[\![\exists x(x = x)]\!]$, which also implies that the tuples have to coincide. Hence, we can conclude that $\pi_A[\![\exists x(x = x)]\!] = \pi_B[\![\exists x(x = x)]\!]$, which is equivalent to $|A| = |B|$, and $\pi_A[\![\vartheta^m_{\bar{c}}(\bar{a})]\!] = \pi_B[\![\vartheta^m_{\bar{c}}(\bar{b})]\!]$. By definition, the latter implies that

$$\sum_{a \in A \setminus \{a_1, \ldots, a_n\}} \pi_A[\![\vartheta^{m-1}_{\bar{c}}(\bar{a}, a)]\!]^{e_{m-1}} = \sum_{b \in B \setminus \{b_1, \ldots, b_n\}} \pi_B[\![\vartheta^{m-1}_{\bar{c}}(\bar{b}, b)]\!]^{e_{m-1}}.$$

Since $\ell > |A| - n = |B| - n$ and $d_{m-1} > \max(\pi_A[\![\vartheta^{m-1}_{\bar{c}}(\bar{a}, a)]\!], \pi_B[\![\vartheta^{m-1}_{\bar{c}}(\bar{b}, b)]\!])$ for all $a \in A \setminus \{a_1, \ldots, a_n\}$ and $b \in B \setminus \{b_1, \ldots, b_n\}$, by lemma 3.46 there is a bijection $h\colon A \setminus \{a_1, \ldots, a_n\} \to B \setminus \{b_1, \ldots, b_n\}$ such that $\pi_A[\![\vartheta^{m-1}_{\bar{c}}(\bar{a}, a)]\!] = \pi_B[\![\vartheta^{m-1}_{\bar{c}}(\bar{b}, h(a))]\!]$ for all $a \in A \setminus \{a_1, \ldots, a_n\}$. With $|A| = |B|$ this implies $\pi_A[\![\chi^{m-1}_{\bar{c}}(\bar{a}, a)]\!] = \pi_B[\![\chi^{m-1}_{\bar{c}}(\bar{b}, h(a))]\!]$ for all $a \in A \setminus \{a_1, \ldots, a_n\}$. Duplicator can win the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ as follows: She provides the bijection h' which extends h to the domain A by $h'(a_i) = b_i$ for all $1 \leq i \leq n$. W.l.o.g. we can assume that Spoiler picks some $a \in A \setminus \{a_1, \ldots, a_n\}$, because if he wins $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, then he must have a winning strategy where he only chooses distinct elements in case the bijections Duplicator provides respect the previous choices, as ensured by h'. We obtain for the updated position $(\bar{a}, a, \bar{b}, h(a))$ that $\pi_A[\![\chi^{m-1}_{\bar{c}}(\bar{a}, a)]\!] = \pi_B[\![\chi^{m-1}_{\bar{c}}(\bar{b}, h(a))]\!]$ must hold. By induction hypothesis, Duplicator has a strategy to win the remaining subgame $BG_{m-1}(\pi_A, \bar{a}, \pi_B, h(a))$. $\square$

Corollary 3.48. Let π_A, π_B be (finite) $\mathbb{N}$-interpretations. If Spoiler wins the game $G_m(\pi_A, \pi_B)$, then $\pi_A \not\equiv_m \pi_B$. If he wins $G(\pi_A, \pi_B)$, it holds that $\pi_A \not\equiv \pi_B$.

Hence, we can infer that m-equivalence between $\mathbb{N}$-interpretations is characterized by the m-turn bijection game. Furthermore, we can use this result in order to show that m-equivalence with regard to the polynomial semiring $\mathbb{N}[X]$ is captured by BG_m as well. Since $\mathbb{N}[X]$ does not admit infinitary operations either for the same reasons we observed for $\mathbb{N}$, we consider finite $\mathbb{N}[X]$-interpretations only. Due to the universal property of $\mathbb{N}[X]$, each variable assignment $\alpha\colon X \to \mathbb{N}$ can be uniquely extended to a homomorphism from $\mathbb{N}[X]$ to $\mathbb{N}$. Hence, if we transform given $\mathbb{N}[X]$-interpretations π_A and π_B into $\mathbb{N}$-interpretations using such a homomorphism and show that the resulting $\mathbb{N}$-interpretations can be separated with some formula of quantifier rank m, then this formula must separate π_A from π_B as well according to the fundamental property. In order to show the separability of the $\mathbb{N}$-interpretations, we can use the bijection game. However, in general we cannot infer that Spoiler wins $BG_m(h \circ \pi_A, h \circ \pi_B)$ if he wins $BG_m(\pi_A, \pi_B)$, as his winning strategy might rely on some equality which is not preserved by h. Hence, h needs to be injective on $\text{img}(\pi_A) \cup \text{img}(\pi_B)$, in order to draw this conclusion. The following lemma illustrates that we can always find a suitable variable assignment which ensures injectivity of the associated homomorphism on the image of both π_A and π_B. A proof of the lemma can be found in [GM21].

Lemma 3.49. Let $\mathbb{N}[X](c, e) \subseteq \mathbb{N}[X]$ denote the set of polynomials with coefficients less than c and exponents smaller than e. There is a variable assignment $\alpha\colon X \to \mathbb{N}$ inducing a homomorphism $h\colon \mathbb{N}[X] \to \mathbb{N}$ whose restriction $h_{|\mathbb{N}[X](c,e)}$ is a bijection from $\mathbb{N}[X](c, e)$ to $\{0, \dots, c^{e^{|X|}} - 1\}$.

Theorem 3.50. Let $c, e, c_2, m \in \mathbb{N}$. For all (finite) $\mathbb{N}[X]$-interpretations π_A, π_B and $\bar{a} \in A^n$, $\bar{b} \in B^n$ such that $\text{img}(\pi_A) \cup \text{img}(\pi_B) \subseteq \mathbb{N}[X](c, e)$ and $|A| < c_2$, $|B| < c_2$, the following are equivalent:

(1) Duplicator wins $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$

(2) $\pi_A[\![\chi^m_{(c_1,c_2)}(\bar{a})]\!] = \pi_B[\![\chi^m_{(c_1,c_2)}(\bar{b})]\!]$ where $c_1 := c^{e^{|X|}}$

(3) $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$

Proof. Following the same reasoning as in 3.47, it suffices to prove (2) $\Rightarrow$ (1). Let $h\colon \mathbb{N}[X] \to \mathbb{N}$ be a homomorphism according to lemma 3.49. Due to the fundamental property, (2) implies $(h \circ \pi_A)[\![\chi^m_{\bar{c}}(\bar{a})]\!] = (h \circ \pi_B)[\![\chi^m_{\bar{c}}(\bar{b})]\!]$. Further, it must hold that $\max \text{img}(h \circ \pi_A) \cup \text{img}(h \circ \pi_A) < c_1$, because of the assumption $\text{img}(\pi_A) \cup \text{img}(\pi_B) \subseteq \mathbb{N}[X](c, e)$ and lemma 3.49. By theorem 3.47, this implies that Duplicator has a winning strategy for $BG_m(h \circ \pi_A, \bar{a}, h \circ \pi_B, \bar{b})$. The strategy ensures that any reachable position $(\bar{a}, a_{n+1}, \dots, a_{n+m}, \bar{b}, b_{n+1}, \dots, b_{n+m})$ induces a local isomorphism between π_A and π_B, i.e., for each $L(\bar{x}) \in \mathrm{Lit}_{n+m}(\tau)$ it holds that $h \circ \pi_A(L(a_1, \dots, a_{n+m})) = h \circ \pi_B(L(b_1, \dots, b_{n+m})))$. Due to injectivity of $h_{|\mathbb{N}[X](c,e)}$, we can derive $\pi_A(L(a_1, \dots, a_{n+m})) = \pi_B(L(b_1, \dots, b_{n+m}))$ for all $L(\bar{x}) \in \mathrm{Lit}_{n+m}(\tau)$. Hence, the strategy must also be winning for Duplicator in the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. $\square$

Note that the results we obtain for the semirings $\mathbb{N}$ and $\mathbb{N}[X]$, which presuppose finiteness of the universes, cannot be lifted to infinite interpretations by substituting the semirings with $\mathbb{N}^\infty$ and $\mathbb{N}^\infty_\infty[\![X]\!]$. In section 3.1.4, we have shown that there are elementarily equivalent $\mathbb{N}^\infty$-interpretations, in which the additional element ∞ does not even occur, such that Spoiler wins the game G_1. As the counterexample can also be construed as a pair of $\mathbb{N}^\infty_\infty[\![X]\!]$-interpretations, we additionally obtain for $\mathbb{N}^\infty_\infty[\![X]\!]$ that first-order logic with semiring semantics is not expressive enough such that m-separability follows from BG_m, or even from G_m.

3.3 Overview

As the previous sections illustrate, the expressive power of first-order logic with semiring semantics and thus, the transferability of the classical Ehrenfeucht-Fraïssé games to $\mathcal{K}$-interpretations, varies significantly with the considered semiring. Therefore, we will give a brief summary of the main observations concerning the applicability of the game G_m and its variants.

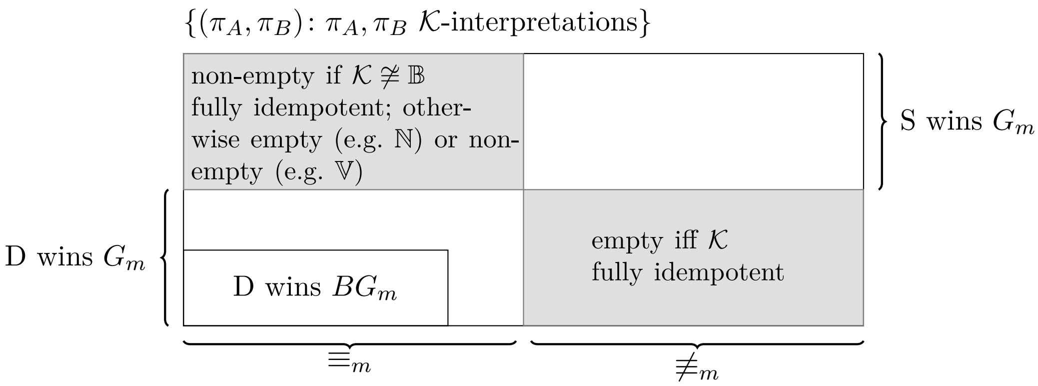


Figure 3.15: Partition of the set of pairs of $\mathcal{K}$-interpretations for a fixed semiring $\mathcal{K}$ based on m-equivalence and the outcome of the m-turn Ehrenfeucht-Fraïssé game G_m, and the relationship of the bijection game BG_m to the resulting classes.

Generally, we have shown that the game G_m does not capture m-equivalence of $\mathcal{K}$-interpretations in any semiring $\mathcal{K}$ which actually constitutes a generalization of classical semantics, so which is not isomorphic to $\mathbb{B}$. Subsequently, we analyzed more precisely for which semirings G_m serves as a proof method for m-equivalence, and in what cases m-separability follows from G_m. Concerning the soundness of G_m, we have seen that Duplicator winning G_m implies m-equivalence exactly for those semirings which are fully idempotent. The implications with respect to the individual semirings we derived based on this result are highlighted in blue in Figure 3.16. With regard to the soundness of the game G, we identified full idempotence as a sufficient but not necessary condition. Depending on the algebraic properties of the underlying semiring, we observed different behaviors with respect to the applicability of G for proving elementary equivalence, based on which the examined semirings can be classified as follows.

(1) Full idempotence is satisfied. A winning strategy for Duplicator in G_m implies m-equivalence. Examples are given by the class of min-max semirings or the polynomial semiring PosBool$[X]$.

(2) The semiring is n-idempotent for some $n \in \mathbb{N}$, i.e., $\sum_{i\in I} k = \sum_{j\in J} k$ and $\prod_{i\in I} k = \prod_{j\in J} k$ for each $k \in \mathcal{K}$ and all index sets I, J with $|I|, |J| \geq n$. A winning strategy for Duplicator in $G_{n\cdot m}$ implies m-equivalence. In particular, a winning strategy for Duplicator in G implies elementary equivalence. For instance, the semiring $\mathbb{W}[X]$ is $|X|$-idempotent.

(3) Depending on the $\mathcal{K}$-interpretations π_A and π_B in question and a separating sentence φ, there is some $n \in \mathbb{N}$ such that Spoiler can prove the separability of π_A and π_B by φ by picking at most n elements. A winning strategy for Duplicator in G implies elementary equivalence. As an example, the semirings $\mathbb{N}^\infty$ and $\mathbb{S}_\infty[X]$ fall into this class.

(4) A winning strategy for Duplicator in G does not imply elementary equivalence in general. This applies, for instance, to the semirings $\mathbb{V}, \mathbb{L}$ as well as $\mathbb{B}_\infty[\![X]\!]$ and $\mathbb{N}^\infty_\infty[\![X]\!]$.

Compared to the classical Ehrenfeucht-Fraïssé game G_m, the bijection game BG_m impedes the winning of Duplicator such that m-equivalence is implied for any semiring $\mathcal{K}$, independent of full idempotence (✓). Based on the formulation as the counting game, we generalized this observation by limiting the cardinality of the sets to be chosen by κ in κ-idempotent semirings, which do not admit counting beyond κ.

For most of the commonly used semirings, one can construct finite interpretations which are elementarily equivalent but non-isomorphic, which already implies that neither G_m nor G is complete (✗). In fact, there are counterexamples where Spoiler already wins the game G_1 despite elementary equivalence. By contrast, finite $\mathbb{V}$- as well as $\mathbb{N}$- or $\mathbb{N}[X]$-interpretations can be defined up to isomorphism. For the Viterbi semiring, we observed that a winning strategy for Spoiler in G_m does not suffice to deduce m-equivalence nevertheless, as the number of moves Spoiler needs to distinguish the $\mathbb{V}$-interpretations might not match the quantifier rank required to separate them. Yet, finite $\mathbb{V}$-interpretations must be separable if Spoiler wins the game G, while this property cannot be lifted to infinite $\mathbb{V}$-interpretations. As opposed to $\mathbb{V}$, we finally proved for both $\mathbb{N}$ and $\mathbb{N}[X]$ that FO with semiring semantics is expressive enough such that m-equivalence is captured by the m-turn bijection game. However, this property cannot be generalized to infinite interpretations by substituting the semirings with $\mathbb{N}^\infty$ and $\mathbb{N}^\infty_\infty[\![X]\!]$, which admit infinitary operations as opposed to $\mathbb{N}$ and $\mathbb{N}[X]$.

Based on these observations, the applicability of the games G_m, G and BG_m to $\mathcal{K}$-interpretations for common semirings $\mathcal{K}$ can be summarized by the subsequent tables, where the gray cells are supposed to indicate that the implication follows from the finiteness of the $\mathcal{K}$-interpretations.

	min-max	$\mathbb{V}$ / $\mathbb{T}$	$\mathbb{L}$ / $\mathbb{D}$	$\mathbb{N}$	$\mathbb{N}^\infty$
D wins $G_m \Rightarrow \equiv_m$	✓	✗	✗	✗	✗
D wins $G \Rightarrow \equiv$	✓	✗	✗	✓	✓
D wins $BG_m \Rightarrow \equiv_m$	✓	✓	✓	✓	✓
S wins $G_m \Rightarrow \not\equiv_m$	✗	✗	✗	✓	✗
S wins $G \Rightarrow \not\equiv$	✗	✗	✗	✓	✗
S wins $BG_m \Rightarrow \not\equiv_m$	✗	✗	✗	✓	✗

	PosBool$[X]$	$\mathbb{W}[X]$	$\mathbb{S}[X]$	$\mathbb{S}_\infty[X]$	$\mathbb{B}[X]$	$\mathbb{B}_\infty[\![X]\!]$	$\mathbb{N}[X]$	$\mathbb{N}^\infty_\infty[\![X]\!]$
D wins $G_m \Rightarrow \equiv_m$	✓	✗	✗	✗	✗	✗	✗	✗
D wins $G \Rightarrow \equiv$	✓	✓	✓	✓	✓	✗	✓	✗
D wins $BG_m \Rightarrow \equiv_m$	✓	✓	✓	✓	✓	✓	✓	✓
S wins $G_m \Rightarrow \not\equiv_m$	✗	✗	✗	✗	✗	✗	✓	✗
S wins $G \Rightarrow \not\equiv$	✗	✗	✗	✗	✗	✗	✓	✗
S wins $BG_m \Rightarrow \not\equiv_m$	✗	✗	✗	✗	✗	✗	✓	✗

Figure 3.16: Transferability of the classical Ehrenfeucht-Fraïssé games G_m and G and the bijection game BG_m to semiring semantics with regard to common application semirings (above) and polynomial semirings (below).

A final observation to be noted is that the direct adaptation of the classical Ehrenfeucht-Fraïssé game, or variants such as the bijection game, lacks the incorporation of the algebraic operations inherent in the semiring. The winning condition only takes into account equalities of semiring elements and the moves do not reflect algebraic properties of the semirings other than their cardinality, either. However, the properties of addition and multiplication, in particular full idempotence, strongly influence the expressive power of first-order logic with semiring semantics. For instance, we have seen that, pertaining to separability, it might be crucial whether valuations are regarded as natural numbers or elements from a min-max semiring. To formalize this observation, it can be shown that G_m, as well as BG_m, is invariant under any injective mapping into another semiring.

Proposition 3.51. Let π_A, π_B be $\mathcal{K}$-interpretations and $f\colon \mathcal{K} \to \mathcal{L}$ map into a semiring $\mathcal{L}$ such that $f_{|K'}$ where $K' := \operatorname{img}(\pi_A) \cup \operatorname{img}(\pi_B)$ is injective. For the $\mathcal{L}$-interpretations $\pi'_A := f \circ \pi_A$, $\pi'_B := f \circ \pi_B$ and each $m \in \mathbb{N}$, Duplicator wins $G_m(\pi_A, \pi_B)$ if, and only if, she wins $G_m(\pi'_A, \pi'_B)$ and Duplicator wins $BG_m(\pi_A, \pi_B)$ if, and only if, she wins $BG_m(\pi'_A, \pi'_B)$.

Proof. Since the set of possible plays in G_m and BG_m only depends on the universes of the $\mathcal{K}$-interpretations the game is played on, $G_m(\pi_A, \pi_B)$ and $G_m(\pi'_A, \pi'_B)$ as well as $BG_m(\pi_A, \pi_B)$ and $BG_m(\pi'_A, \pi'_B)$ admit the same plays. In order to show that each play is won by the same player in the two games, respectively, it suffices to prove that each position $(\bar{a}, \bar{b})$, where $\bar{a} = (a_1, \dots, a_m) \in A^m$ and $\bar{b} = (b_1, \dots, b_m) \in B^m$, induces a local isomorphism between π_A and π_B if, and only if, it induces a local isomorphism between π'_A and π'_B. By definition, $(\bar{a}, \bar{b})$ induces a local isomorphism between π_A and π_B if, and only if, $\pi_A(L(\bar{a})) = \pi_B(L(\bar{b}))$ for all $L \in \mathrm{Lit}_m(\tau)$. Due to injectivity of the function $f_{|K'}$, this is equivalent to $\pi'_A(L(\bar{a})) = f \circ \pi(L(\bar{a})) = f \circ \pi'_B(L(\bar{b})) = \pi'_B(L(\bar{a}))$ for all $L \in \mathrm{Lit}_m(\tau)$, that is, $(\bar{a}, \bar{b})$ inducing a local isomorphism between π'_A and π'_B. □

What we actually aim at in order to find an appropriate game characterizing m-equivalence with regard to semiring semantics, is invariance under injective homomorphisms instead of arbitrary injective mappings. This is because we obtain with the fundamental property for any sentence $\varphi \in \mathrm{FO}(\tau)$ that

$$\pi_A[\![\varphi]\!] = \pi_B[\![\varphi]\!] \text{ if, and only if, } (h \circ \pi_A)[\![\varphi]\!] = (h \circ \pi_B)[\![\varphi]\!]$$

where h is an injective homomorphism, which implies $\pi_A \equiv_m \pi_B$ if, and only if, $(h \circ \pi_A) \equiv_m (h \circ \pi_B)$. By contrast, the equivalence does not apply to arbitrary injective mappings, as presumed by the games G_m and BG_m.

Chapter 4

Deviating from the Notion of Equivalence

The previous chapter illustrated that the direct translation of the Ehrenfeucht-Fraïssé game to $\mathcal{K}$-interpretations is generally not appropriate to characterize the natural definition of m-equivalence under semiring semantics. Thus, the question arises as to what notion of equivalence is actually characterized by the Ehrenfeucht-Fraïssé game when applying it to $\mathcal{K}$-interpretations. In order to examine this question and gain a better understanding on how the game behaves on $\mathcal{K}$-interpretations, we will consider two alternative notions of equivalence. First, we will analyze whether the game G_m on $\mathcal{K}$-interpretations captures m-equivalence with respect to classical FO when associating classical structures to the $\mathcal{K}$-interpretations by incorporating the semiring within the vocabulary. As a second approach, we define an equivalence term based on a logic which is evaluated on *two-sorted structures* with a second universe for the semiring elements.

4.1 Copying Relation Symbols

In order to formalize on which $\mathcal{K}$-interpretations Duplicator wins the game G_m, an alternative notion of m-equivalence between $\mathcal{K}$-interpretations which resembles classical first-order equivalence but still takes into account semiring valuations is sought. One way to implement this is to associate a classical structure $\mathfrak{A}$ over an extended vocabulary τ^K which consists of multiple "copies" of the original relation symbols in τ with each $\mathcal{K}$-interpretation π_A. In this manner, a valuation $\pi_A(R\bar{a}) = k$ can be encoded in $\mathfrak{A}$ by adding the tuple $\bar{a}$ to the relation $(R_k)^{\mathfrak{A}}$, so instead of a single relation symbol R the vocabulary τ^K needs to contain relation symbols R_k for every semiring element $k \in K$. Note that in this context $\mathfrak{A} \models \neg R_k\bar{a}$ does not reflect that $\pi_A(\neg R\bar{a}) = k$ but $\pi_A(R\bar{a}) \neq k$. This is why additional relation symbols $\overline{R_k}$ are included in τ^K for the purpose of modeling the negated τ-literals' valuations.

Definition 4.1. A $\mathcal{K}$-interpretation $\pi_A\colon \mathrm{Lit}_A(\tau) \to K$ induces the τ^K-structure $\mathfrak{A} = (A, \tau^K)$ where $\tau^K := \{R_k, \overline{R_k}\colon R \in \tau, k \in K\}$ such that $\mathfrak{A} \models R_k\bar{a}$ if, and only if, $\pi_A(R\bar{a}) = k$ and $\mathfrak{A} \models \overline{R_k}\bar{a}$ if, and only if, $\pi_A(\neg R\bar{a}) = k$ for $R \in \tau$ and $k \in K$.

A central difference between m-equivalence of $\mathcal{K}$-interpretations and m-equivalence of their induced τ^K-structures is caused by the direct access to the semiring elements in the logic. Recall, for instance, the introductory $\mathcal{K}_4$-interpretations π_A^2 and π_B^2 where $\mathcal{K}_4$ is a min-max semiring.

π_A^2 :

A	R	$\neg R$
a_1	1	0
a_2	2	0
a_3	4	0

π_B^2 :

B	R	$\neg R$
b_1	1	0
b_2	3	0
b_3	4	0

We observed previously that, although the valuation $\pi_A^2(Ra_2)$ cannot be duplicated with some $b \in B$, neither $\exists x Rx$ nor $\forall x Rx$ separates π_A^2 from π_B^2 and that the $\mathcal{K}_4$-interpretations are in fact 1-equivalent. While this counterexample demonstrates that $\mathrm{FO}(\tau)$ with semiring semantics is not as expressive as presumed by the moves of Spoiler, it does not apply to the $\tau^{\mathcal{K}_4}$-structures $\mathfrak{A}^2$ and $\mathfrak{B}^2$ they induce, which can be separated by $\exists x R_2 x$.

It is noticeable that the induced τ^K-structures do not reflect the semiring operations at all, but neither does the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations, as we observed in the previous chapter. Another crucial observation indicating that the game $G_m(\pi_A, \pi_B)$ might actually capture m-equivalence between the induced τ^K-structures, instead of m-equivalence between the $\mathcal{K}$-interpretations themselves, is that the encoding of the semiring valuations preserves local isomorphisms.

Proposition 4.2. For any two $\mathcal{K}$-interpretations π_A, π_B which induce the τ^K-structures $\mathfrak{A}, \mathfrak{B}$ and elements $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$, the mapping σ defined by $\sigma\colon a_i \mapsto b_i$ for $1 \leq i \leq n$ is a local isomorphism between π_A and π_B if, and only if, σ is a local isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Despite this observation, it is not clear yet in what cases the Ehrenfeucht-Fraïssé theorem can be applied to the induced τ^K-structures, as the extended vocabulary τ^K becomes infinite if the underlying semiring $\mathcal{K}$ is, although we presume finiteness of the vocabulary τ the $\mathcal{K}$-interpretations refer to. However, m-equivalence of the induced τ^K-structures can be derived from G_m if Duplicator wins, independent of the cardinalities of universes and semiring.

Theorem 4.3. Given any two $\mathcal{K}$-interpretations π_A and π_B inducing the τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$, it must hold that $\mathfrak{A} \equiv_m \mathfrak{B}$ if Duplicator wins the game $G_m(\pi_A, \pi_B)$.

Proof. Based on elements $\bar{a} \in A^n$, $\bar{b} \in B^n$ and a formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau^K)$ with $\bar{x} = (x_1, \ldots, x_n)$ separating $(\mathfrak{A}, \bar{a})$ from $(\mathfrak{B}, \bar{b})$, we construct a winning strategy for Spoiler in the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ where $\mathrm{qr}(\varphi(\bar{x})) = m$ by induction over the structure of $\varphi(\bar{x})$.

Case 1. If $\varphi(\bar{x})$ is a literal in $\mathrm{FO}(\tau^K)$, it follows immediately that the mapping σ induced by $\bar{a}$ and $\bar{b}$ cannot be a local isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. By proposition 4.2, σ cannot be a local isomorphism between π_A and π_B, either. Hence, the winning condition is violated and Spoiler wins $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. If $\varphi(\bar{x}) = \psi(\bar{x}) \circ \vartheta(\bar{x})$, where $\circ \in \{\vee, \wedge\}$ and $\mathrm{qr}(\varphi(\bar{x})) = m$, $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ must be separable by $\psi(\bar{x})$ or $\vartheta(\bar{x})$. Since $\mathrm{qr}(\psi(\bar{x})) \leq m$ and $\mathrm{qr}(\vartheta(\bar{x})) \leq m$, applying the induction hypothesis yields that Spoiler wins $G_{m'}(\pi_A, \bar{a}, \pi_B, \bar{b})$ for some $m' \leq m$, hence he wins the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ as well.

Case 3. If $\varphi(\bar{x}) = \neg\psi(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) = m$, we can infer that $\psi(\bar{x})$ separates $(\mathfrak{A}, \bar{a})$ from $(\mathfrak{B}, \bar{b})$ as well and apply the induction hypothesis, which yields that Spoiler wins $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 4. Let $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$, where $\mathrm{qr}(\varphi(\bar{x})) = m$ and assume w.l.o.g. that $\mathfrak{A} \models \exists x \psi(\bar{a}, x)$. Spoiler can pick some $a \in A$ such that $\mathfrak{A} \models \psi(\bar{a}, a)$. Since $\varphi(\bar{x})$ is separating by assumption, it must hold that $\mathfrak{B} \not\models \psi(\bar{b}, b)$ for all possible answers $b \in B$ of Duplicator. We have that $\mathrm{qr}(\psi(\bar{x})) = m-1$, so it follows from the induction hypothesis that Spoiler wins the remaining game $G_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$.

Because of the logical equivalence $\forall x \psi(\bar{x}, x) \equiv \neg\exists x \neg\psi(\bar{x}, x)$ in $\mathrm{FO}(\tau^K)$ with classical semantics, the case $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$ can be omitted. $\square$

By contrast, the cardinality of the vocabulary impacts the completeness G_m. However, when constructing characteristic sentences, we can make use of the fact that we do not consider arbitrary classical structures but only those that are induced by $\mathcal{K}$-interpretations. For any relation symbol $R \in \tau$ which is part of the vocabulary of some $\mathcal{K}$-interpretation π_A and any tuple $\bar{a}$ of elements, there is exactly one $k \in K$ such that $\mathfrak{A} \models R_k \bar{a}$, whereas $\mathfrak{A} \not\models R_\ell \bar{a}$ for all $\ell \neq k$. Hence, if the universe of π_A is finite, then only finitely many relations $R_k \in \tau^K$ are non-empty, even though τ^K might be infinite. Therefore, it suffices if one of the universes or the semiring is finite in order to infer m-separability of $\mathfrak{A}$ and $\mathfrak{B}$ based on a winning strategy of Spoiler in $G_m(\pi_A, \pi_B)$.

Theorem 4.4. If A, B or K is finite, then a winning strategy for Spoiler in $G_m(\pi_A, \pi_B)$ implies $\mathfrak{A} \not\equiv_m \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are the τ^K-structures induced by π_A and π_B.

Proof. By symmetry, we can assume that A or K is finite. By induction on $m \in \mathbb{N}$, we construct a characteristic formula $\chi^m_{\pi_A,\bar{a}}(x_1, \ldots, x_n) \in \mathrm{FO}(\tau^K)$ of quantifier rank m for any $\bar{a} = (a_1, \ldots, a_n) \in A^n$ such that $\mathfrak{B} \models \chi^m_{\pi_A,\bar{a}}(\bar{b})$ ensures for each $\bar{b} \in B^n$ that Duplicator has a winning strategy for $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. In order to prove the finiteness of $\chi^m_{\pi_A,\bar{a}}(x_1, \ldots, x_n)$ we additionally show that for finite K, there are only finitely many different formulae $\chi^m_{\pi_A,\bar{a}}(x_1, \ldots, x_n)$ for varying $\bar{a}$ of fixed length n. Let $\mathrm{Lit}^+_n(\tau)$ denote the set of unnegated and $\mathrm{Lit}^-_n(\tau)$ the set of negated τ-literals with variables from $\{x_1, \ldots, x_n\}$. For $m = 0$ and $\bar{x} = (x_1, \ldots, x_n)$ let

$$\begin{aligned}\chi^0_{\pi_A,\bar{a}}(\bar{x}) := \varphi^=_{\bar{a}}(\bar{x}) \wedge \bigwedge\{R_k x_{i_1} \ldots x_{i_r} | R x_{i_1} \ldots x_{i_r} \in \mathrm{Lit}^+_n(\tau), \pi_A(R a_{i_1} \ldots a_{i_r}) = k\} \wedge \\ \bigwedge\{\overline{R_k} x_{i_1} \ldots x_{i_r} | \neg R x_{i_1} \ldots x_{i_r} \in \mathrm{Lit}^-_n(\tau), \pi_A(\neg R a_{i_1} \ldots a_{i_r}) = k\},\end{aligned}$$

with $\varphi^=_{\bar{a}}(\bar{x})$ defining the equalities between the components of $\bar{a}$ according to

$$\varphi^=_{\bar{a}}(\bar{x}) := \bigwedge_{\substack{1 \le i < j \le n: \\ a_i = a_j}} x_i = x_j \wedge \bigwedge_{\substack{1 \le i < j \le n: \\ a_i \neq a_j}} x_i \neq x_j.$$

As the sets $\mathrm{Lit}^+_n(\tau)$ and $\mathrm{Lit}^-_n(\tau)$ are finite, the conjunctions are built over finite sets, hence $\chi^0_{\pi_A,\bar{a}}(\bar{x})$ is well-defined. Further, this ensures that there are only finitely many distinct formulae $\chi^0_{\pi_A,\bar{a}}(\bar{x})$ for varying $\bar{a} \in A^n$ if K is finite. It readily follows from the definition of $\chi^0_{\pi_A,\bar{a}}(\bar{x})$ that $\mathfrak{B} \models \chi^0_{\pi_A,\bar{a}}(\bar{b})$ implies that the mapping σ induced by $\bar{a}$ and $\bar{b}$ is a bijection and that it holds that $\pi_A(L(\bar{a})) = \pi_B(L(\bar{b}))$ for all $L(\bar{x}) \in \mathrm{Lit}_n(\tau)$. Hence, σ must be a local isomorphism between π_A and π_B and Duplicator wins $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$. We inductively define $\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x})$ as

$$\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x}) := \bigwedge_{a \in A} \exists x\ \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x) \wedge \forall x \bigvee_{a \in A} \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x).$$

In case A is finite, conjunction and disjunction refer only to finitely many elements, which ensures the finiteness of the formula. Otherwise, K must be finite by assumption. The set $\{\chi^m_{\pi_A,\bar{a},a}(\bar{x}, x) \colon a \in A\} \subseteq \{\chi^m_{\pi_A,\bar{a},a}(\bar{x}, x) \colon (\bar{a}, a) \in A^{n+1}\}$ is finite by induction hypothesis, which ensures the finiteness of the formula $\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x})$ and of the set $\{\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x}) \colon \bar{a} \in A^n\}$. To prove the correctness of the formula, suppose that $\mathfrak{B} \models \chi^{m+1}_{\pi_A,\bar{a}}(\bar{b})$. By construction of $\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x})$ this implies that

(1) for all $a \in A$ there is some $b \in B$ such that $\mathfrak{B} \models \chi^m_{\pi_A,\bar{a},a}(\bar{b}, b)$ and

(2) for all $b \in B$ there is some $a \in A$ such that $\mathfrak{B} \models \chi^m_{\pi_A,\bar{a},a}(\bar{b}, b)$.

From applying the induction hypothesis it follows that

(1) for all $a \in A$ there is some $b \in B$ such that Duplicator wins the game $G_m(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$ and

(2) for all $b \in B$ there is some $a \in A$ such that Duplicator wins the game $G_m(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$,

which is equivalent to Duplicator winning $G_{m+1}(\pi_A, \bar{a}, \pi_B, \bar{b})$. □

In general, the characteristic formulae $\chi^m_{\pi_A,\bar{a}}(\bar{x})$ we just constructed are not well-defined if both A and K are infinite. However, it remains to verify whether the finiteness of one of universes or the semiring is indeed a necessary condition for $\mathfrak{A} \not\equiv_m \mathfrak{B}$ to be implied by Spoiler winning $G_m(\pi_A, \pi_B)$. For general classical structures $\mathfrak{A}$ and $\mathfrak{B}$ over infinite vocabulary which do not have to be induced by $\mathcal{K}$-interpretations, not even G_1 can be used to show that $\mathfrak{A} \not\equiv_1 \mathfrak{B}$. As an example, consider classical structures $\mathfrak{A}$ and $\mathfrak{B}$ over vocabulary $\{R_i \colon i \in \mathbb{N}\}$ consisting of unary relation symbols, where the universes are defined by $A = \{a_i \colon i \in \mathbb{N}\}$ and $B = \{b_i \colon i \in \mathbb{N}\} \cup \{b\}$. The relation symbols are interpreted according to the following rule: $\mathfrak{A} \models R_i a_j$ if, and only if, $i = j$ and, analogously, $\mathfrak{B} \models R_i b_j$ if, and only if, $i = j$. For the additional element $b \in B$ it holds that $\mathfrak{B} \not\models R_i b$ for all $i \in \mathbb{N}$. Clearly, Spoiler wins $G_1(\mathfrak{A}, \mathfrak{B})$ by picking $b \in B$, which cannot be duplicated in $\mathfrak{A}$. However, it holds that $\mathfrak{A} \equiv \mathfrak{B}$ because $\mathfrak{A} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0$ for every finite $\tau_0 \subseteq \tau$. Certainly, we are only interested in those classical structures which are induced by $\mathcal{K}$-interpretations, which changes the situation. As the atomic properties of each element in π_A and π_B are determined by finitely many relations in τ^K, the existence of a separating sentence is ensured if Spoiler wins in the case $m = 1$.

Proposition 4.5. For any two $\mathcal{K}$-interpretations π_A and π_B, it must hold that $\mathfrak{A} \not\equiv_1 \mathfrak{B}$ if Spoiler has a winning strategy for $G_1(\pi_A, \pi_B)$, even if A, B and K are infinite.

Proof. Fix a winning strategy of Spoiler in $G_1(\pi_A, \pi_B)$ and assume w.l.o.g. that he chooses an element $a \in A$. Based on this element, consider the sentence $\exists x \chi^0_{\pi_A,a}(x)$ where $\chi^0_{\pi_A,a}(x)$ is defined according to the proof of theorem 4.4. The finiteness of $\chi^0_{\pi_A,a}(x)$ is ensured, independent of the cardinality of A, B and K, as $\mathrm{Lit}_1(\tau)$ is finite. Since Spoiler wins $G_1(\pi_A, \pi_B)$ by picking $a \in A$, for every $b \in B$ it must hold that $\pi_A(Ra \dots a) \neq \pi_B(Rb \dots b)$ or $\pi_A(\neg Ra \dots a) \neq \pi_B(\neg Rb \dots b)$ for some $R \in \tau$.

This implies that $\mathfrak{B} \not\models \chi^0_{\pi_A,a}(b)$ for all $b \in B$. Hence, the sentence $\exists x \chi^0_{\pi_A,a}(x)$ separates $\mathfrak{A}$ from $\mathfrak{B}$. $\square$

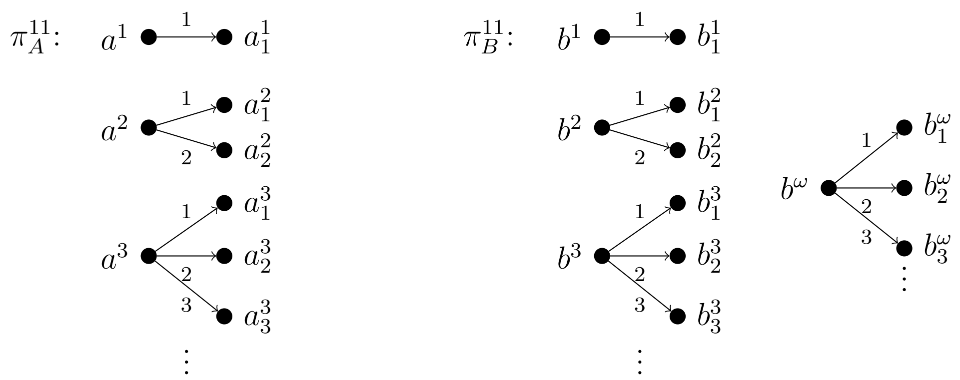
Note that the method in the previous proof cannot be transferred to quantifier rank 2, as infinitely many different answers of Duplicator in the first turn would have to be taken into account. Indeed, for quantifier ranks larger than 1, the existence of a separating sentence is not ensured if Spoiler wins the game G_m in the case of A, B and K being infinite.

Proposition 4.6. There are $\mathcal{K}$-interpretations π_A and π_B, where A, B and K are infinite and Spoiler wins $G_2(\pi_A, \pi_B)$ while $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. We construct $\mathbb{N}^\infty$-interpretations π^{11}_A, π^{11}_B over the vocabulary $\tau = \{E\}$ consisting of a single binary relation symbol. The universes are given by

$$A := \{a^n : n \in \mathbb{N}_{>0}\} \cup \{a^n_i : n, i \in \mathbb{N}_{>0}, i \leq n\}$$
$$B := \{b^n : n \in \mathbb{N}_{>0}\} \cup \{b^n_i : n, i \in \mathbb{N}_{>0}, i \leq n\} \cup \{b^\omega\} \cup \{b^\omega_i : i \in \mathbb{N}_{>0}\}$$

and the valuations satisfy $\pi^{11}_A(Ea^n a^n_i) = \pi^{11}_B(Eb^n b^n_i) = i$ for all $a^n, a^n_i \in A$ and $b^n, b^n_i \in B$. The corresponding negated $\{E\}$-literals over A and B are mapped to 0, while the remaining positive $\{E\}$-literals over A and B are valuated with 0 and their negations with 1.



Spoiler has the following winning strategy for the game $G_2(\pi^{11}_A, \pi^{11}_B)$. First, he chooses b^ω. If Duplicator answers with a^n for some $n \in \mathbb{N}_{>0}$, then Spoiler picks b^ω_{n+1} in the second round. For any possible answer $a \in A$ of Duplicator, it holds that $\pi^{11}_A(Ea^n a) < n + 1 = \pi^{11}_B(Eb^\omega b^\omega_{n+1})$, so Spoiler wins. Otherwise, Duplicator answered with some $a^n_i \in A$ in the first turn. In this case, Spoiler picks b^ω_1, which results in $\pi^{11}_A(Ea^n_i a) = 0 \neq 1 = \pi^{11}_B(Eb^\omega b^\omega_1)$ for each possible answer $a \in A$ in the second turn.

Let $\mathfrak{A}^{11}$ and $\mathfrak{B}^{11}$ be the $\tau^{\mathbb{N}^\infty}$-structures induced by π_A^{11} and π_B^{11}. In order to prove that $\mathfrak{A}^{11} \equiv \mathfrak{B}^{11}$, it suffices to show that Duplicator has a winning strategy for $G(\mathfrak{A}^{11} \restriction \tau_0, \mathfrak{B}^{11} \restriction \tau_0)$ for any finite $\tau_0 \subseteq \tau^{\mathbb{N}^\infty}$.

Let $\tau_0 \subseteq \tau^{\mathbb{N}^\infty}$ and $k \in \mathbb{N}$ be maximal such that $E_k \in \tau_0$. Based on k, we define partitions

$$\begin{aligned}\mathcal{P}_A^k :=& \{A^n \colon n \in \mathbb{N}_{>0}\} \cup \{A_i^n \colon n, i \in \mathbb{N}_{>0}, i \leq \min(n, k+1)\} \text{ and}\\ \mathcal{P}_B^k :=& \{B^n \colon n \in \mathbb{N}_{>0}\} \cup \{B_i^n \colon n, i \in \mathbb{N}_{>0}, i \leq \min(n, k+1)\}\\ &\cup \{B^\omega\} \cup \{B_i^\omega \colon i \in \mathbb{N}, i \leq k+1\}\end{aligned}$$

of A and B according to $A^n = \{a^n\}$, $A_i^n = \{a_i^n\}$ if $i \leq \min(n,k)$ as well as $A_{k+1}^n = \{a_{k+i}^n \colon i \in \mathbb{N}_{>0}, k+i \leq n\}$ for each $n \in \mathbb{N}_{>0}$ with $k < n$. The sets B^n and B_i^n are defined analogously, while $B^\omega = \{b^\omega\}$, $B_i^\omega = \{b_i^\omega\}$ if $i \leq k$ and $B_{k+1}^\omega = \{b_{k+i}^\omega \colon i \in \mathbb{N}_{>0}\}$. Depending on the number m of moves Spoiler picks in the first step, we construct a bijection $f_{m,k} \colon \mathcal{P}_A^k \to \mathcal{P}_B^k$ according to

$$f_{m,k}(A^n) = \begin{cases} B^n, & n < m+k \\ B^\omega, & n = m+k \\ B^{n-1}, & n > m+k \end{cases} \quad \text{and} \quad f_{m,k}(A_i^n) = \begin{cases} B_i^n, & n < m+k \\ B_i^\omega, & n = m+k \\ B_i^{n-1}, & n > m+k \end{cases},$$

which ensures $|A'| = |f_{m,k}(A')|$, or both $|A'| \geq m$ and $|f_{m,k}(A')| \geq m$ for all $A' \in \mathcal{P}_A^k$. Hence, in the game $G_m(\mathfrak{A}^{11} \restriction \tau_0, \mathfrak{B}^{11} \restriction \tau_0)$, Duplicator is able to duplicate each $a \in A'$ with an arbitrary $b \in f_{m,k}(A')$ and each $b \in B'$ with some $a \in f_{m,k}^{-1}(B')$ by making sure that equalities are maintained. $\square$

Summing up, the game G_m played on $\mathcal{K}$-interpretations π_A and π_B characterizes m-equivalence between the induced τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$ in case at least one of the universes or the underlying semiring is finite. If both universes as well as the semiring is infinite, a winning strategy for Duplicator in $G_m(\pi_A, \pi_B)$ implies $\mathfrak{A} \equiv_m \mathfrak{B}$, whereas the converse implication in general only applies to quantifier rank 1.

4.1.1 Application to the Counting Game

We can derive from the previous result that m-equivalence of $\mathcal{K}$-interpretations π_A and π_B is, in general, not implied by m-equivalence between the induced τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$, although the translation into $\mathfrak{A}$ and $\mathfrak{B}$ allows direct access to the semiring elements. For instance, we can infer concerning the introductory $\mathbb{N}$-interpretations π_A^1, π_B^1 that the $\tau^{\mathbb{N}}$-structures $\mathfrak{A}^1$ and $\mathfrak{B}^1$ they induce are 1-equivalent, as Duplicator wins $G_1(\pi_A^1, \pi_B^1)$, even though $\pi_A^1 \not\equiv_1 \pi_B^1$. Our intuition is that the additional expressive power semiring semantics provides comprises counting of certain semiring valuations. Hence, we now aim to formalize that the additional expressive power does not exceed counting. To this end, we apply the m-turn counting game and make use of the induced τ^K-structures in order to establish a reference to a logic including a counting mechanism. In its original formulation, the k-pebble counting game captures the k-variable fragment of FO with counting quantifiers on finite structures. Thus, we will verify that m-equivalence of the τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$ in FO with counting quantifiers, which we refer to as FOC, is ensured if Duplicator wins the game $CG_m(\pi_A, \pi_B)$ on finite $\mathcal{K}$-interpretations π_A, π_B, as introduced in section 3.2. Further, we will examine whether the finiteness of the $\mathcal{K}$-interpretations is actually required in this context. Having proven that a winning strategy for Duplicator in $CG_m(\pi_A, \pi_B)$ yields $\pi_A \equiv_m \pi_B$, we can then infer that $\pi_A \equiv_m \pi_B$ is implied by m-equivalence of $\mathfrak{A}$ and $\mathfrak{B}$ with regard to FOC.

Definition 4.7. The syntax and semantics of the logic FOC(τ) coincides with FO(τ), except for the formula building rule

- $\exists x \varphi(\bar{x}, x) \in \mathrm{FO}(\tau)$ if $\varphi(\bar{x}, x) \in \mathrm{FO}(\tau)$,

which is replaced by

- $\exists^{\geq i} x \varphi(\bar{x}, x) \in \mathrm{FOC}(\tau)$ if $\varphi(\bar{x}, x) \in \mathrm{FOC}(\tau)$ for each $i \in \mathbb{N}$.

Accordingly, we write $(A, \tau) \models \exists^{\geq i} x \varphi(\bar{a}, a)$ if, and only if, there are distinct elements $a_1, \ldots, a_i \in A$ such that $(A, \tau) \models \varphi(\bar{a}, a_j)$ for all $1 \leq j \leq i$.

For $i \in \mathbb{N}$ we denote the fragment of FOC(τ) which only contains quantifiers $\exists^{\geq i'}$ with $i' \leq i$ as $\mathrm{FOC}_i(\tau)$. Note that as only finite formulae are permitted, elementary equivalence with regard to FOC(τ) coincides with elementary equivalence in FO(τ), since each formula $\exists^{\geq i} x \varphi(\bar{x}, x)$ can be translated into a logically equivalent first-order formula by nesting quantifiers. However, FOC(τ) enables us to express properties while using less nested quantifiers, hence when considering formulae of fixed quantifier rank more structures can be distinguished in FOC(τ) than in FO(τ).

As mentioned before, our goal is to prove that $\mathfrak{A} \equiv_m^{\mathrm{FOC}(\tau^K)} \mathfrak{B}$ implies that Duplicator wins $CG_m(\pi_A, \pi_B)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are the τ^K-structures induced by π_A and π_B. More precisely, we will show for each $j \in \mathbb{N}$ that Duplicator wins $CG_m^j(\pi_A, \bar{a}, \pi_B, \bar{b})$ on finite $\mathcal{K}$-interpretations π_A, π_B if $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC_j}(\tau^K)} (\mathfrak{B}, \bar{b})$ for the induced τ^K-structures $\mathfrak{A}, \mathfrak{B}$. In particular, this implies that Duplicator wins $CG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ if $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}(\tau^K)} (\mathfrak{B}, \bar{b})$, because on finite $\mathcal{K}$-interpretations, Duplicator wins CG_m if, and only if, she wins CG_m^j for all $j \in \mathbb{N}$. Since only finitely many valuations occur in π_A and π_B in case the universes are finite, it does not have to be assumed that the semiring is finite. Even though the vocabulary

τ^K becomes infinite if the underlying semiring is, it suffices to take into account a finite subset $\tau^{K'} \subseteq \tau^K$ which is induced by the semiring elements that actually occur in the $\mathcal{K}$-interpretations of interest, as the following lemma illustrates.

Lemma 4.8. Let π_A and π_B be $\mathcal{K}$-interpretations with induced τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$. For the vocabulary $\tau^{K'}$ induced by $K' := \mathrm{img}(\pi_A) \cup \mathrm{img}(\pi_B)$ and each $j \in \mathbb{N}$, $\bar{a} \in A^n$ and $\bar{b} \in B^n$ it holds that

$$(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^K)} (\mathfrak{B}, \bar{b}) \text{ if, and only if, } (\mathfrak{A} \restriction \tau^{K'}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B} \restriction \tau^{K'}, \bar{b}).$$

Proof. Clearly, $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^K)} (\mathfrak{B}, \bar{b})$ implies $(\mathfrak{A} \restriction \tau^{K'}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B} \restriction \tau^{K'}, \bar{b})$, as $\tau^{K'} \subseteq \tau^K$. To prove the converse implication, we associate a formula $\varphi^*(\bar{x})$ of $\mathrm{FOC}_j(\tau^{K'})$ with each $\varphi(\bar{x}) \in \mathrm{FOC}_j(\tau^K)$ such that $\mathfrak{A} \models \varphi^*(\bar{a})$ if, and only if, $\mathfrak{A} \models \varphi(\bar{a})$ and, analogously, $\mathfrak{B} \models \varphi^*(\bar{b})$ if, and only if, $\mathfrak{B} \models \varphi(\bar{b})$ for all tuples $\bar{a}$ and $\bar{b}$ of elements in A and B. For each $R \in \tau^K \setminus \tau^{K'}$, let $\varphi^*(\bar{x}) := x_{i_1} \neq x_{i_1}$ if $\varphi(\bar{x}) = Rx_{i_1} \dots x_{i_r}$. For all remaining atoms $\varphi(\bar{x})$ we set $\varphi^*(\bar{x}) := \varphi(\bar{x})$ and inductively construct $\varphi^*(\bar{x})$ for complex formulae $\varphi(\bar{x})$ via $(\neg\psi)^*(\bar{x}) := \neg\psi^*(\bar{x})$, $(\psi \circ \vartheta)^*(\bar{x}) := \psi^*(\bar{x}) \circ \vartheta^*(\bar{x})$ for $\circ \in \{\vee, \wedge\}$ and $(Qx\psi(\bar{x}, x))^*(\bar{x}) := Qx\psi^*(\bar{x}, x)$ where $Q \in \{\forall\} \cup \{\exists^{\geq i} \colon i \in \mathbb{N}\}$. The correctness of the constructed formula $\varphi^*(\bar{x})$ readily follows by induction due to the fact that $R^{\mathfrak{A}} = R^{\mathfrak{B}} = \varnothing$ for all $R \in \tau^K \setminus \tau^{K'}$. Hence, we obtain that $\varphi(\bar{x})$ separates $(\mathfrak{A}, \bar{a})$ from $(\mathfrak{B}, \bar{b})$ if, and only if, $\varphi^*(\bar{x})$ is separating with regard to $(\mathfrak{A} \restriction \tau^{K'}, \bar{a})$ and $(\mathfrak{B} \restriction \tau^{K'}, \bar{b})$. $\square$

Theorem 4.9. Let π_A and π_B be finite $\mathcal{K}$-interpretations inducing the τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$. Given any $j, m \in \mathbb{N}$ and tuples $\bar{a} \in A^n$, $\bar{b} \in B^n$, Duplicator wins $CG_m^j(\pi_A, \bar{a}, \pi_B, \bar{b})$ if $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^K)} (\mathfrak{B}, \bar{b})$.

Proof. We prove the claim by induction on $m \in \mathbb{N}$. For the base case $m = 0$ suppose that $(\mathfrak{A}, \bar{a}) \equiv_0^{\mathrm{FOC}_j(\tau^K)} (\mathfrak{B}, \bar{b})$. This immediately implies that $\bar{a}$ and $\bar{b}$ induce a local isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ and thus also between π_A and π_B, which causes Duplicator to win $CG_0^j(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Assume that the claim is true for some arbitrary, fixed $m \in \mathbb{N}$. As π_A and π_B are finite by assumption, the set $K' := \mathrm{img}(\pi_A) \cup \mathrm{img}(\pi_B)$ must be finite as well. Suppose that $(\mathfrak{A}, \bar{a}) \equiv_{m+1}^{\mathrm{FOC}_j(\tau^K)} (\mathfrak{B}, \bar{b})$, which implies $(\mathfrak{A}', \bar{a}) \equiv_{m+1}^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B}', \bar{b})$ where $\mathfrak{A}'$ and $\mathfrak{B}'$ are the $\tau^{K'}$-reducts of $\mathfrak{A}$ and $\mathfrak{B}$. Let X be the set Spoiler chooses in the first move of the game $CG_{m+1}^j(\pi_A, \bar{a}, \pi_B, \bar{b})$. W.l.o.g suppose that $X \subseteq A$ and let $X_1, \dots, X_\ell$ be the coarsest partition of X such that for all $a, a' \in X_i$ it holds that $(\mathfrak{A}', \bar{a}, a) \equiv_m (\mathfrak{A}', \bar{a}, a')$. For $1 \leq i \leq \ell$ let $\Phi_i^{m,j} \subseteq \mathrm{FOC}_j(\tau^{K'})$ be the set of formulae $\varphi(\bar{x}, x)$ of quantifier rank at most m such that $(\mathfrak{A}', \bar{a}, a) \models \varphi(\bar{x}, x)$ for all $a \in X_i$. Since $\tau^{K'}$ is finite and relational, there are only finitely many formulae in $\Phi_i^{m,j}$ up to logical equivalence. We fix a maximal subset $\Psi_i^{m,j} \subseteq \Phi_i^{m,j}$ which does not contain any two logically equivalent formulae. By definition we obtain

$$\mathfrak{A}' \models \exists^{\geq |X_i|} x \bigwedge_{\psi \in \Psi_i^{m,j}} \psi(\bar{a}, x).$$

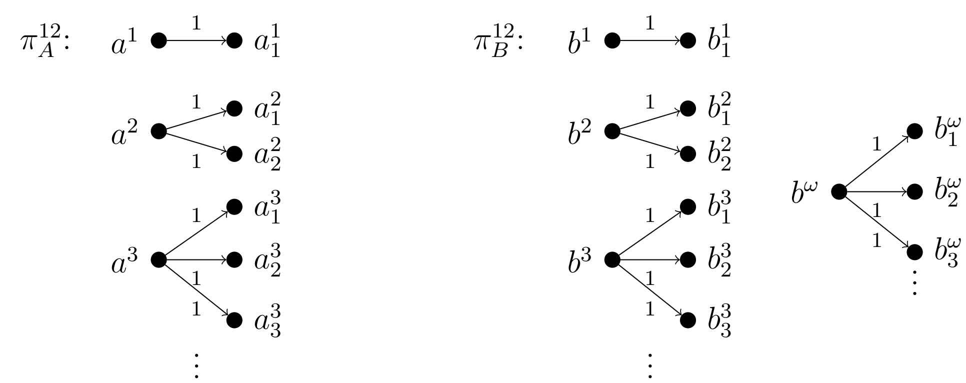
As $|X_i| \leq j$ for each $1 \leq i \leq \ell$ and $(\mathfrak{A}', \bar{a}) \equiv_{m+1}^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B}, \bar{b})$, this implies

$$\mathfrak{B}' \models \exists^{\geq |X_i|} x \bigwedge_{\psi \in \Psi_i^{m,j}} \psi(\bar{b}, x).$$

Hence, there must be $b_1, \dots b_{|X_i|} \in B$ such that $\mathfrak{B}' \models \psi(\bar{b}, b_k)$ for all $\psi(\bar{x}, x) \in \Psi_i^{m,j}$ and $1 \leq k \leq |X_i|$. For $Y_i := \{b_1 \dots b_{|X_i|}\}$ we obtain $(\mathfrak{A}', \bar{a}, a) \equiv_m^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B}', \bar{b}, b)$ for all $a \in X_i$ and $b \in Y_i$. To win $CG_{m+1}^j(\pi_A, \bar{a}, \pi_B, \bar{b})$, Duplicator can respond $Y := \bigcup_{1 \leq i \leq \ell} Y_i$ to X. It holds that $|X| = |Y|$, because for $i \neq i'$ we must have that $Y_i \cap Y_{i'} = \varnothing$. The element $b \in Y$ Spoiler chooses afterwards must be in some set Y_i and Duplicator can answer with some arbitrary $a \in X_i$. For the updated position $(\bar{a}, a, \bar{b}, b)$ it holds that $(\mathfrak{A}', \bar{a}, a) \equiv_m^{\mathrm{FOC}_j(\tau^{K'})} (\mathfrak{B}', \bar{b}, b)$. By lemma 4.8, this implies $(\mathfrak{A}, \bar{a}, a) \equiv_m^{\mathrm{FOC}_j(\tau^{K})} (\mathfrak{B}, \bar{b}, b)$ such that the application of the induction hypothesis yields that Duplicator wins the remaining subgame $CG_m^j(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$. □

The fact that for finite $\mathcal{K}$-interpretations $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}_j(\tau^{K})} (\mathfrak{B}, \bar{b})$ implies that Duplicator wins $CG_m^j(\pi_A, \bar{a}, \pi_B, \bar{b})$ suggests that $(\mathfrak{A}, \bar{a}) \equiv_m^{\mathrm{FOC}(\tau^{K})} (\mathfrak{B}, \bar{b})$ might ensure the winning of Duplicator in $CG_m^\omega(\pi_A, \bar{a}, \pi_B, \bar{b})$ for infinite $\mathcal{K}$-interpretations as well. However, this is not true due to the finiteness of the formulae in $\mathrm{FOC}(\tau^K)$, which can be observed in the following counterexample.

Example 4.10. Consider the $\mathbb{B}$-interpretations π_A^{12} and π_A^{12} which can be derived from the previously considered $\mathbb{N}^\infty$-interpretations π_A^{11} and π_B^{11} by applying the mapping $f\colon \mathbb{N} \to \mathbb{B}$ with $n \mapsto 1$ if, and only if, $n > 0$.



Spoiler wins the game $CG_2^\omega(\pi_A^{12}, \pi_B^{12})$ as follows. First, he picks the set $\{b^\omega\}$. If Duplicator answers with some $\{a_i^n\}$, he chooses $\{b_1^\omega\}$ in the second turn. As there is no vertex $a \in A$ such that $\pi_A(Ea_i^n a) = 1$, Spoiler wins. Hence, Duplicator must answer with some set $\{a^n\}$ in the first round. But then, Spoiler can pick the set $\{b_1^\omega, \dots, b_{n+1}^\omega\}$ afterwards. As there are only n distinct nodes $a \in A$ such that $\pi_A(Ea^n a) = 1$, Duplicator cannot find a suitable answer and loses the play in this case as well. However, for the induced $\tau^{\mathbb{B}}$-structures $\mathfrak{A}^{12}$ and $\mathfrak{B}^{12}$ we have that $\mathfrak{A}^{12} \equiv^{\mathrm{FOC}(\tau^{\mathbb{B}})} \mathfrak{B}^{12}$, because Duplicator wins the game $G(\mathfrak{A}^{12}, \mathfrak{B}^{12})$ by playing analogously to the strategy for the $\mathbb{B}_\infty[\![X]\!]$-interpretations π_A^8 and π_B^8 described in the proof of proposition 3.34.

Yet, we can derive the desired upper bound on the expressive power of first-order logic with semiring semantics for finite $\mathcal{K}$-interpretations, as a winning strategy for Duplicator in $CG_m(\pi_A, \pi_B)$ ensures $\pi_A \equiv_m \pi_B$ for any two $\mathcal{K}$-interpretations π_A and π_B. It illustrates that the additional expressive power semiring semantics provides in contrast to classical FO on the induced τ^K-structures does not exceed counting. As for j-idempotent semirings $\pi_A \equiv_m \pi_B$ is implied by Duplicator winning $CG_m^j(\pi_A, \pi_B)$, we can further conclude that j-idempotent semirings do not admit counting beyond j.

Corollary 4.11. Given any two finite $\mathcal{K}$-interpretations π_A and π_B inducing the τ^K-structures $\mathfrak{A}$ and $\mathfrak{B}$, it holds that $\pi_A \equiv_m^{\mathrm{FO}(\tau)} \pi_B$ if $\mathfrak{A} \equiv_m^{\mathrm{FOC}(\tau^K)} \mathfrak{B}$. If $\mathcal{K}$ is j-idempotent for some $j \in \mathbb{N}$, then $\pi_A \equiv_m^{\mathrm{FO}(\tau)} \pi_B$ is implied by $\mathfrak{A} \equiv_m^{\mathrm{FOC}_j(\tau^K)} \mathfrak{B}$.

A final observation to be noted is that the converse implication of theorem 4.9 can be easily proven by induction, which provides further insights into m-equivalence of $\mathbb{N}$-interpretations, as CG_m captures m-equivalence of (finite) $\mathbb{N}$-interpretations.

Proposition 4.12. Given any two (finite) $\mathbb{N}$-interpretations π_A and π_B with induced $\tau^{\mathbb{N}}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, it holds that $\mathfrak{A} \equiv_m^{\mathrm{FOC}(\tau^K)} \mathfrak{B}$ if, and only if, $\pi_A \equiv_m^{\mathrm{FO}(\tau)} \pi_B$.

In particular, this result is interesting in the context of canonical counting interpretations, which were introduced in [GT17a]. With each finite τ-structure $\mathfrak{A}$, one can associate an $\mathbb{N}$-interpretation $\pi_{\#\mathfrak{A}}$, the *canonical counting interpretation* for $\mathfrak{A}$, with the same universe A and vocabulary τ such that for $L \in \mathrm{Lit}_{\mathrm{A}}(\tau)$ it holds that $\pi_{\#\mathfrak{A}}(L) = 1$ if $\mathfrak{A} \models L$ and $\pi_{\#\mathfrak{A}}(L) = 0$ otherwise. In [GT17a], it was shown that for each sentence $\varphi \in \mathrm{FO}(\tau)$, the valuation $\pi_{\#\mathfrak{A}}[\![\varphi]\!]$ provides the number of proof trees witnessing $\mathfrak{A} \models \varphi$. Note that the $\tau^{\mathbb{N}}$-structure induced by $\pi_{\#\mathfrak{A}}(L)$ does not coincide with $\mathfrak{A}$, as it relies on an extended vocabulary. However, it can be easily shown that the $\tau^{\mathbb{N}}$-structures induced by some canonical counting interpretations $\pi_{\#\mathfrak{A}}$ and $\pi_{\#\mathfrak{B}}$ are m-equivalent in $\mathrm{FOC}(\tau)$, if and only if, the underlying

τ-structures satisfy $\mathfrak{A} \equiv_m^{\mathrm{FOC}(\tau)} \mathfrak{B}$. Hence, proposition 4.12 yields, in particular, that for any two finite τ-structures $\mathfrak{A}$, $\mathfrak{B}$, it holds that $\mathfrak{A} \equiv_m^{\mathrm{FOC}(\tau)} \mathfrak{B}$ if, and only if, $\pi_{\#\mathfrak{A}} \equiv_m^{\mathrm{FO}(\tau)} \pi_{\#\mathfrak{B}}$ where $\pi_{\#\mathfrak{A}}$ and $\pi_{\#\mathfrak{B}}$ are the canonical counting interpretations for $\mathfrak{A}$ and $\mathfrak{B}$.

4.2 Two-sorted structures

Within the previous section, we observed that associating a classical structure over an extended vocabulary with each $\mathcal{K}$-interpretation yields a notion of equivalence which is captured by the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations, provided that one of the universes or the semiring is finite. However, it is noticeable that only little information about the semiring is taken into account when translating a $\mathcal{K}$-interpretation into a τ^K-structure. In fact, it is only the cardinality of the semiring $\mathcal{K}$ which actually affects the resulting classical structure, whereas the operations are not taken into account in any way. This raises the question whether the expressive power is increased if additional information about the semiring such as the natural order or the operations are accessible in the logic. Further, it is not clear whether the incorporation of such relations and function may substitute the direct access to the semiring elements in the logic such that the resulting notion of equivalence is still captured by the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations.

In order to examine this question, we deviate from regarding classical structures and make use of *two-sorted structures*, which are based on the notion of *metafinite structures* introduced in [GG98] and provide a more flexible framework to incorporate a second sort of elements with a separate vocabulary. A metafinite structure $\mathfrak{D} = (\mathfrak{A}, \mathfrak{R}, W)$ consists of two individual structures $\mathfrak{A}$ and $\mathfrak{R}$ which are interlinked by a set W of weight functions mapping tuples of elements in the *primary structure* $\mathfrak{A}$ into the *secondary structure* $\mathfrak{R}$. Thereby, the primary structure $\mathfrak{A}$ is assumed to be finite, as the consideration of metafinite structures aims at generalizing the methods from finite model theory. As derived in the previous chapter, we may omit the requirement that the universes must be finite in our context in case the semiring is, which is why we do not demand the finiteness of the primary structure a priori. Moreover, we do not incorporate a separate vocabulary in the primary structure but model the interpretation of the literals using weight functions which map into the secondary structure, the semiring. The functions and relations of the secondary structure can be used to incorporate, for instance, the natural order, the operations, or semiring elements as constants.

Definition 4.13. A $\mathcal{K}$-interpretation $\pi_A\colon \mathrm{Lit}_A(\tau_U) \to K$ with a structure (K, τ_S) induces the *two-sorted structure* $\mathfrak{D}_A = (A, (K, \tau_S), \{w_R, w_{\neg R}\colon R \in \tau_U\})$ where the weight functions $w_R, w_{\neg R} \colon A^{\text{arity}(R)} \to K$ are defined by $w_R(\bar{a}) = \pi_A(R\bar{a})$ and $w_{\neg R}(\bar{a}) = \pi_A(\neg R\bar{a})$ for all $\bar{a} \in A^{\text{arity}(R)}$.

In order to incorporate the secondary structure as well as the weight functions into first-order logic, we distinguish two different kinds of terms, which we refer to as *U-terms* and *S-terms*. As their notation suggests, they are evaluated with elements of the universe or of the semiring, respectively.

Definition 4.14. The set of U-terms and S-terms with respect to the vocabularies τ_U and τ_S is inductively defined according to the following rules.

(1) Each variable is a *U-term*.

(2) If $t_U^1, \dots, t_U^n$ are U-terms and R is an n-ary relation symbol in τ_U, then $w_R(t_U^1, \dots, t_U^n)$ and $w_{\neg R}(t_U^1, \dots, t_U^n)$ are *S-terms*.

(3) If $t_S^1, \dots, t_S^n$ are S-terms and f is an n-ary function symbol in τ_S, then $f(t_S^1, \dots, t_S^n)$ is an S-term as well. In particular, each constant in τ_S is an S-term.

Based on the two sorts of terms, we define the set $\text{FO}(\tau_U, \tau_S)$ of formulae in *two-sorted FO* as the smallest set closed under rules $(1) - (3)$.

(1) If t_1 and t_2 are U-terms or S-terms, then $t_1 = t_2$ is a formula in $\text{FO}(\tau_U, \tau_S)$.

(2) If $t_S^1, \dots, t_S^n$ are S-terms and R is an n-ary relation symbol in τ_S, then $Rt_S^1, \dots, t_S^n$ is in $\text{FO}(\tau_U, \tau_S)$.

(3) For each $\varphi, \psi \in \text{FO}(\tau_U, \tau_S)$, the formulae $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ as well as $\exists x\varphi$ and $\forall x\varphi$, where x is a variable, are contained in $\text{FO}(\tau_U, \tau_S)$.

The formulae in $\text{FO}(\tau_U, \tau_S)$ are interpreted by a two-sorted structure $\mathfrak{D}$ with a suitable variable assignment $\beta \colon X \to A$. The semantics of U-terms is determined by β according to $[\![x]\!]^{(\mathfrak{D},\beta)} := \beta(x) \in A$, while the semantics of S-terms is defined inductively by

$$[\![w_L(t_U^1, \dots, t_U^n)]\!]^{(\mathfrak{D},\beta)} := w_L([\![t_U^1]\!]^{(\mathfrak{D},\beta)}, \dots, [\![t_U^n]\!]^{(\mathfrak{D},\beta)}) \text{ and}$$
$$[\![f(t_S^1, \dots, t_S^n)]\!]^{(\mathfrak{D},\beta)} := f([\![t_S^1]\!]^{(\mathfrak{D},\beta)}, \dots, [\![t_S^n]\!]^{(\mathfrak{D},\beta)})$$

where $L \in \{R, \neg R \colon R \in \tau_U\}$ and $f \in \tau_S$. The semantics of the formulae in $\text{FO}(\tau_U, \tau_S)$ follows from the semantics of U- and S-terms, analogous to classical FO. As before, we use the notation $\mathfrak{D} \models \varphi(a_1, \dots, a_n)$ to denote $(\mathfrak{D}, \beta) \models \varphi(x_1, \dots, x_n)$ with $\beta(x_i) = a_i$ for $1 \le i \le n$.

Note that the quantifiers are restricted to range over elements of the primary structure's universe only. We now aim to analyze the expressive power of $\mathrm{FO}(\tau_U, \tau_S)$ on two-sorted structures induced by $\mathcal{K}$-interpretations and a fixed secondary structure (K, τ_S). Hence, the two-sorted structures we examine only interpret the weight functions but not the relation and function symbols in τ_S. The basic definitions such as quantifier rank, m-equivalence and elementary equivalence in $\mathrm{FO}(\tau_U, \tau_S)$ are analogous to classical first-order logic.

First of all, it is noticeable that moving from $\mathcal{K}$-interpretations to two-sorted structures allows the separation of structures, although the underlying $\mathcal{K}$-interpretations are elementarily equivalent. Consider for instance the secondary structure $(K, \leq)$, where $\leq$ is interpreted by the order induced by addition in $\mathcal{K}$, and recall the $\mathcal{K}_3$-interpretations π^3_A and π^3_B with $\mathcal{K}_3$ being a min-max semiring.

π^3_A :

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_1	1	3	0	0
a_2	2	1	0	0
a_3	3	2	0	0

π^3_B :

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	3	1	0	0
b_2	1	2	0	0
b_3	2	3	0	0

In spite of elementary equivalence of π^3_A and π^3_B, the associated two-sorted structures $\mathfrak{D}^3_A$ and $\mathfrak{D}^3_B$ with secondary part $(\{0, 1, 2, 3\}, \leq)$ can be separated. For the sentence

$$\varphi := \exists x \exists y (x \neq y \wedge w_{R_2}(x) \leq w_{R_1}(x) \wedge w_{R_2}(y) \leq w_{R_1}(y))$$

we obtain $\mathfrak{D}^3_A \models \varphi$, whereas $\mathfrak{D}^3_B \not\models \varphi$, as the transformation from π^3_A into π^3_B does not respect the natural order.

Hence, the question arises as to what functions and relations need to be contained in the secondary structure (K, τ_S) such that the resulting equivalence term is captured by the game G_m on $\mathcal{K}$-interpretations. Using the subsequent lemma, it can be shown that it does not suffice if addition and multiplication, the neutral elements as well as the natural order is available in the secondary structure in order to separate all two-sorted structures induced by $\mathcal{K}$-interpretations on which Spoiler wins the Ehrenfeucht-Fraïssé game.

Lemma 4.15. Let π be a $\mathcal{K}$-interpretation and K' be the closure of $\mathrm{img}(\pi) \cup \{0, 1\}$ under addition and multiplication. Further, let $h \colon \mathcal{K} \to \mathcal{K}$ be an endomorphism such that $h_{|K'}$ is injective and respects the order induced by addition, i.e., $k \leq \ell$ if, and only if, $h(k) \leq h(\ell)$ for all $k, \ell \in K'$. For each $\bar{a} \in A^n$, it holds that $(\mathfrak{D}, \bar{a}) \equiv (\mathfrak{D}', \bar{a})$ where $\mathfrak{D}$ and $\mathfrak{D}'$ are the two-sorted structures associated with π and $\pi' = h \circ \pi$ with secondary part $(K, +, \cdot, 0, 1, \leq)$ corresponding to the components of $\mathcal{K}$ and the order induced by addition.

Proof. Let $\bar{a} \in A^n$. Since h is an endomorphism, it follows by induction that $[\![t(\bar{a})]\!]^{\mathfrak{D}'} = h([\![t(\bar{a})]\!]^{\mathfrak{D}})$ for all S-terms $t(\bar{x})$. For equalities of U-terms $\varphi(\bar{x}) = x_i = x_j$ it clearly holds that $\mathfrak{D} \models \varphi(\bar{a})$ if, and only if, $\mathfrak{D}' \models \varphi(\bar{a})$, as the free variables are instantiated with $\bar{a}$ in both cases. If $\varphi(\bar{x}) = t_1 \circ t_2$ for S-terms t_1, t_2 and $\circ \in \{=, \leq\}$ it holds that $[\![t_i(\bar{a})]\!]^{\mathfrak{D}} \in K'$ where $i \in \{1, 2\}$. Due to injectivity of $h_{|K'}$ and the compatibility with $\leq$ this implies $\mathfrak{D} \models \varphi(\bar{a})$ if, and only if, $\mathfrak{D}' \models \varphi(\bar{a})$. For complex formulae $\varphi(\bar{x})$ which are built according to formula building rule (3), the equivalence follows by induction. Hence, we can conclude that $(\mathfrak{D}, \bar{a}) \equiv (\mathfrak{D}', \bar{a})$. ☐

We can apply this condition to the two-sorted structures $\mathfrak{D}_A^2$ and $\mathfrak{D}_B^2$ associated with the introductory $\mathcal{K}_4$-interpretations π_A^2 and π_B^2 and the secondary part $(\{0, 1, 2, 3, 4\}, +, \cdot, 0, 4, \leq)$.

π_A^2 :

A	R	$\neg R$
a_1	1	0
a_2	2	0
a_3	4	0

π_B^2 :

B	R	$\neg R$
b_1	1	0
b_2	3	0
b_3	4	0

It holds that $\pi_B^2 = h \circ h_A^2$ where $h \colon \mathcal{K}_4 \to K_4$ with $h \colon x \mapsto x$ for $x \in \{0, 1, 3, 4\}$ and $h \colon 2 \mapsto 3$. Since h respects the underlying linear order and preserves the neutral elements, h is an endomorphism. Moreover, $h_{|K'}$ where $K' := \{0, 1, 2, 4\}$ is injective. Hence, with lemma 4.15 we can conclude that $\mathfrak{D}_A^2 \equiv \mathfrak{D}_B^2$. Note that as opposed to $\mathfrak{D}_A^2$ and $\mathfrak{D}_B^2$, π_A^2 and π_B^2 can be separated with a sentence of quantifier rank 2, so in this case the outcome of the game G corresponds to elementary equivalence in semiring semantics rather than to elementary equivalence of the associated two-sorted structures.

Proposition 4.16. There are $\mathcal{K}$-interpretations π_A and π_B such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\mathfrak{D}_A \equiv \mathfrak{D}_B$ for the two-sorted structures associated with π_A and π_B with secondary part $(K, +, \cdot, 0, 1, \leq)$ consisting of the components of $\mathcal{K}$ and the natural order.

This observation gives rise to including the semiring elements as constants in the secondary structure in order to increase the expressive power of two-sorted FO. In this manner, two-sorted FO resembles classical FO on the induced τ^K-structures, however it comes with formulae of the form $w_{R_1}(\bar{x}) = w_{R_2}(\bar{x})$, which are not available in FO(τ^K). Even if the secondary structure contains additional functions and relations enabling formulae such as $w_{R_1}(\bar{x}) + w_{R_2}(\bar{x}) \leq w_{R_3}(\bar{x})$, the expressive power is not increased and likewise captured by the game G_m on the underlying $\mathcal{K}$-interpretations if one of the universes or the semiring is finite.

Theorem 4.17. Let π_A, π_B be $\mathcal{K}$-interpretations such that A, B or K is finite. The following are equivalent for each $m \in \mathbb{N}$.

(1) Duplicator wins $G_m(\pi_A, \pi_B)$

(2) $\mathfrak{D}_A \equiv_m \mathfrak{D}_B$ where $\mathfrak{D}_A$ and $\mathfrak{D}_B$ are the two-sorted structures induced by π_A and π_B with secondary part $(K, \{c_k \colon k \in K\})$ interpreting each c_k by k

(3) $\mathfrak{D}'_A \equiv_m \mathfrak{D}'_B$ for all two-sorted structures $\mathfrak{D}'_A, \mathfrak{D}'_B$ associated with π_A and π_B and secondary part (K, τ_S) such that $\{c_k \colon k \in K\} \subseteq \tau_S$ with each c_k interpreted by k

Proof. Clearly, it holds that (3) $\Rightarrow$ (2). By symmetry, we can assume that A or K is finite. We show (2) $\Rightarrow$ (1) by constructing a characteristic sentence $\chi^m_{\pi_A,\bar{a}}$ for each $m \in \mathbb{N}$ and $\bar{a} \in A^n$. For $\bar{x} = (x_1, \ldots, x_n)$ let

$$\chi^0_{\pi_A,\bar{a}}(\bar{x}) := \varphi^{=}_{\bar{a}}(\bar{x}) \wedge \bigwedge \{w_L(x_{i_1} \ldots x_{i_r}) = c_{\pi_A(La_{i_1} \ldots a_{i_r})} | Lx_{i_1} \ldots x_{i_r} \in \mathrm{Lit}_n(\tau)\},$$

where $\varphi^{=}_{\bar{a}}(\bar{x})$ describes the equalities and inequalities of the components of $\bar{a}$ as in the proof of theorem 4.4. If K is finite, then the set $\{\chi^0_{\pi_A,\bar{a}}(\bar{x}) \colon \bar{a} \in A^n\}$ must also be finite, as there are only finitely many literals in $\mathrm{Lit}_n(\tau)$. By induction, this ensures that $\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x})$ given by

$$\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x}) := \bigwedge_{a \in A} \exists x \; \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x) \wedge \forall x \bigvee_{a \in A} \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x)$$

is finite and thus a formula of $\mathrm{FO}(\tau_U, \tau_S)$. Analogous to the reasoning in 4.4, it follows by induction that Duplicator wins $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ if $\mathfrak{D}_B \models \chi^m_{\pi_A,\bar{a}}(\bar{b})$ for some $\bar{b} \in B^n$.

It remains to prove (1) $\Rightarrow$ (3). Let $\varphi(\bar{x}) \in \mathrm{FO}(\tau_S, \tau_U)$ with $\{c_k \colon k \in K\} \subseteq \tau_S$ be a formula of quantifier rank m separating $(\mathfrak{D}'_A, \bar{a})$ and $(\mathfrak{D}'_B, \bar{b})$ for some $\bar{a} \in A^n$ and $\bar{b} \in B^n$. It can be shown by induction on the structure of $\varphi(\bar{x})$ that Spoiler has a winning strategy for $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 1. If $\varphi(\bar{x}) = x_i = x_j$ corresponds to an equality of U-terms, there cannot be a bijection mapping $\bar{a}$ to $\bar{b}$. Hence, the current position does not induce a local isomorphism from π_A to π_B and Spoiler wins $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

Case 2. If $\varphi(\bar{x}) = t^1_S(\bar{x}) = t^2_S(\bar{x})$ or $\varphi(\bar{x}) = Rt^1_S(\bar{x}) \ldots t^n_S(\bar{x})$ where each $t^i_S(\bar{x})$ is an S-term, then there must be an S-term $t^i_S(\bar{x})$ such that $[\![t^i_S(\bar{a})]\!]^{\mathfrak{D}'_A} \neq [\![t^i_S(\bar{b})]\!]^{\mathfrak{D}'_B}$. As the descent to proper subterms is well-founded, there must be an S-term $w_L(\bar{x})$ such that $[\![w_L(\bar{a})]\!]^{\mathfrak{D}'_A} \neq [\![w_L(\bar{b})]\!]^{\mathfrak{D}'_B}$. Consequently, $\bar{a}$ and $\bar{b}$ do not correspond to a local isomorphism between π_A and π_B and Spoiler wins $G_0(\pi_A, \bar{a}, \pi_B, \bar{b})$.

All remaining cases can be transferred from the proof of theorem 4.3. $\square$

Chapter 5

Modifications of the Game Rules

After examining to what extent the Ehrenfeucht-Fraïssé game as well as its variants, the bijection and counting game, can be generalized towards semiring semantics and what notion of equivalence is actually captured by the Ehrenfeucht-Fraïssé game on $\mathcal{K}$-interpretations, we will now discuss certain modifications of the game rules in order to account for the problems observed previously. First, we will quantify to what extent the individual semiring elements admit counting and discuss an approach to incorporate the measure into the game rules. Afterwards, we derive a game relying on homomorphisms into the Boolean semiring which characterizes m-equivalence of $\mathcal{K}$-interpretations for distributive lattices.

5.1 Counting in Semirings

In [GG98], Grädel and Gurevich propose a variant of the Ehrenfeucht-Fraïssé game for metafinite structures with a fixed secondary part. It captures a calculus of terms which are evaluated by elements of the secondary structure, similar as first-order formulae are evaluated with semiring elements in semiring semantics. Despite the similarities, every multiset operation on the secondary universe is accessible in the term calculus which makes it very expressive and does not capture the essence of semiring semantics, where two fixed operations, whose algebraic properties may vary, determine the semantics of the formulae.

In particular, the multiset operations allow counting to the full extent. While this assumption is justified for semiring semantics with respect to complex semirings such as $\mathbb{N}$ or $\mathbb{N}[X]$, the ability to count is not exhausted in various other semirings. As we have seen earlier, the influence the number of $a \in A$ such that $\pi_A[\![\psi(\bar{a}, x)]\!] = k$ has on the semantics of $\exists x \psi(\bar{x}, x)$ and $\forall x \psi(\bar{x}, x)$ may vary for different $k \in \mathcal{K}$. In section 3.2, we formalized this observation using the notion of κ-idempotence, which allows us to restrict the cardinalities of the sets chosen in the counting game. However, κ does not refer to individual semiring elements but describes the semiring as a whole. Hence, the question arises whether a measure of idempotence can be derived for the single semiring elements such that occurrences of 1 in the $\mathbb{V}$-interpretations π_A^9, π_B^9 and π_C^9, for instance, are treated differently compared to the occurrences of 0.5.

$$\pi_A^9 : \begin{array}{c||c|c} A & R & \neg R \\ \hline\hline a_1 & 1 & 0 \\ \hline a_2 & 1 & 0 \\ \hline a_3 & 0.5 & 0 \end{array} \quad \equiv_1 \quad \pi_B^9 : \begin{array}{c||c|c} B & R & \neg R \\ \hline\hline b_1 & 1 & 0 \\ \hline b_2 & 0.5 & 0 \end{array} \quad \not\equiv_1 \quad \pi_C^9 : \begin{array}{c||c|c} C & R & \neg R \\ \hline\hline c_1 & 1 & 0 \\ \hline c_2 & 0.5 & 0 \\ \hline c_3 & 0.5 & 0 \end{array}$$

In order to implement this, we will modify the game on metafinite structures introduced in [GG98]. To this end, we quantify to what extent a semiring element can influence sums and products by the number of occurrences by making use of the following lemma, which applies to naturally ordered and *square comparable* semirings $\mathcal{K}$, where $k \leq k^2$ or $k^2 \leq k$ for all $k \in K$. To simplify notation, we write $n * k$ for $n \in \mathbb{N}$ and $k \in \mathcal{K}$ in order to denote $\sum_{1 \leq i \leq n} k$.

Lemma 5.1. Let $\mathcal{K}$ be a naturally ordered and square comparable. For each $n_0, m_0 \in \mathbb{N}$ with $n_0 < m_0$ such that $n_0 * k = m_0 * k$, it holds that $n_0 * k = m * k$ for all $m \geq n_0$. Analogously, $k^{n_0} = k^{m_0}$ implies $k^{n_0} = k^m$ or all $m \geq n_0$.

Proof. Let $n_0 < m_0 \in \mathbb{N}$ such that $n_0 * k = m_0 * k$. This immediately implies that $n_0 * k = (n_0 + i(m_0 - n_0)) * k$ for all $i \in \mathbb{N}$. Since $n_0 < m_0$ by assumption, for any $m \in \mathbb{N}$ with $m > n_0$, it holds that

$$n_0 * k \leq m * k \leq (n_0 + m(m_0 - n_0)) * k = n_0 * k.$$

Analogously, for $n_0 < m_0 \in \mathbb{N}$ with $k^{n_0} = k^{m_0}$ we can conclude for all $i \in \mathbb{N}$ that $k^{n_0} = k^{n_0 + i(m_0 - n_0)}$. By assumption, it must hold that $k \leq k^2$ or $k^2 \leq k$, so we have that $k^n \leq k^m$ for all $n \leq m$ or $k^n \leq k^m$ for all $n \geq m$. This implies for all $m \geq n_0$ that

$$\begin{aligned} k^{n_0} &\leq k^m \leq k^{n_0 + m(m_0 - n_0)} = k^{n_0} \text{ or} \\ k^{n_0} &= k^{n_0 + m(m_0 - n_0)} \leq k^m \leq k^{n_0}. \end{aligned}$$

In both cases we obtain that $k^{n_0} = k^m$ due to antisymmetry of the natural order, which proves the claim. $\square$

Definition 5.2. Let $\mathcal{K}$ be a naturally ordered and square comparable. We associate with each $k \in \mathcal{K}$ the unique cardinal number $i(k) \leq \omega$ such that

(1) $i(k) = \omega$ if $m * k \neq n * k$ or $k^m \neq k^n$ for all $m, n \in \mathbb{N}_{>0}$

(2) $i(k) * k = n * k$ and $k^{i(k)} = k^n$ for all $n \in \mathbb{N}_{>0}$ with $n \geq i(k)$,

(3) $i(k) * k \neq n * k$ or $k^{i(k)} \neq k^n$ for all $n \in \mathbb{N}_{>0}$ with $n < i(k)$

As an example, we obtain for the Viterbi semiring $i(1) = 1$ and $i(0.5) = \omega$, whereas $i(x + y) = 2$ in $\mathbb{W}[\{x, y\}]$. Based on the derived measure $i(k)$, we construct the game $CG'_m(\pi_A, \pi_B)$ for $\mathcal{K}$-interpretations π_A and π_B where $\mathcal{K}$ is naturally ordered, square comparable and ω-idempotent as follows.

Definition 5.3. In each play of $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ where $\bar{a} = (a_1, \ldots, a_n) \in A^n$ and $\bar{b} = (b_1, \ldots, b_n) \in B^n$, the following steps are repeated m times.

(1) Spoiler chooses a function $f \colon X \to K$ where $X \in \{A, B\}$.

(2) Duplicator answers with a function $g \colon Y \to K$ where Y is the universe Spoiler did not choose such that $|f^{-1}(k)| = |g^{-1}(k)|$, or $|f^{-1}(k)| \geq i(k)$ and $|g^{-1}(k)| \geq i(k)$ for each $k \in K$.

(3) Spoiler chooses some element $y \in Y$.

(4) Duplicator answers with an element $x \in X$ such that $f(x) = g(y)$ and the pair (x, y) is added to the current position.

Duplicator wins the play, if, and only if, the resulting position, which we denote as $(a_1, \ldots, a_{n+m}, b_1, \ldots, b_{n+m})$, corresponds to a local isomorphism from π_A to π_B.

Theorem 5.4. If Duplicator wins the game $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ on $\mathcal{K}$-interpretations π_A, π_B with $\mathcal{K}$ being naturally ordered, square comparable and ω-idempotent, it holds that $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$.

Proof. We construct a winning strategy for Spoiler in the game $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ where $\bar{a} \in A^n$ and $\bar{b} \in B^n$ based on a formula $\varphi(\bar{x})$ which separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. Thereby, we only consider the case $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ where $Q \in \{\exists, \forall\}$, since all remaining cases are analogous to the proof of theorem 3.40. As $\varphi(\bar{x})$ is assumed to be separating, it holds that

$$\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \text{ or}$$
$$\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!].$$

In both cases Spoiler can choose the mapping $f\colon A \to K$ defined by $a \mapsto \pi_A[\![\psi(\bar{a}, a)]\!]$. Towards a contradiction, assume that Duplicator answers with $g\colon B \to K$ where $b \mapsto \pi_B[\![\psi(\bar{b}, b)]\!]$ for all $b \in B$. According to the game rules, this would imply for each $k \in K$ that

(1) $|\{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}| = |\{b \in B\colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}|$ or

(2) $|\{a \in A\colon \pi_A[\![\psi(\bar{a}, a)]\!] = k\}| \geq i(k)$ and $|\{b \in B\colon \pi_B[\![\psi(\bar{b}, b)]\!] = k\}| \geq i(k)$.

But due to the definition of $i(k)$ and ω-idempotence, this would yield

$$\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] \text{ and}$$
$$\prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!],$$

which constitutes a contradiction. Hence, there must be some $b \in B$ such that $g(b) \neq \pi_B[\![\psi(\bar{b}, b)]\!]$. Spoiler picks this element b and Duplicator has to respond with some $a \in A$ such that $f(a) = \pi_A[\![\psi(\bar{a}, a)]\!] = g(b)$, so for the resulting position $(\bar{a}, a, \bar{b}, b)$ we have that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq \pi_B[\![\psi(\bar{b}, b)]\!]$. From applying the induction hypothesis we can infer that Spoiler wins $CG'_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$. $\square$

Although the game CG'_m constitutes a sound proof method for m-equivalence, its weakness is that the game rules do not ensure that the mapping Spoiler chooses refers to valid valuation of some formula. In the game $CG'_1(\pi_A^9, \pi_B^9)$, for instance, he can choose the mapping $f\colon a \mapsto 0.5$ for each $a \in A$ such Duplicator has to provide a unique duplicate in B to each element in A. For simple examples such as $CG'_1(\pi_A^9, \pi_B^9)$ this can be circumvented by demanding that the mapping Spoiler provides must correspond to the valuations of some literal. However, this is not sufficient for multiple turns and $\mathcal{K}$-interpretations with several relations of higher arity.

Proposition 5.5. If K is infinite or $i(k) = \omega$ for some $k \in K$, then the game $CG'_m(\pi_A, \pi_B)$ is equivalent to $CG^\omega_m(\pi_A, \pi_B)$.

In case K is finite and $i(k)$ is a natural number for each $k \in \mathcal{K}$, the game $CG'_m(\pi_A, \pi_B)$ facilitates the winning of Duplicator compared to $CG^\omega_m(\pi_A, \pi_B)$. More precisely, it can be shown that Duplicator wins $CG'_m(\pi_A, \pi_B)$ not more frequently than $CG^{i(\mathcal{K})-1}_m(\pi_A, \pi_B)$ and not less frequently than she wins $CG^{i(\mathcal{K})}_m(\pi_A, \pi_B)$ where $i(\mathcal{K}) := \sum_{k \in \mathcal{K}} i(k)$.

Theorem 5.6. Let K be finite and $i(k) < \omega$ for all $k \in K$. If Duplicator wins $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, she also wins $CG^{i(\mathcal{K})-1}_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. If Duplicator wins $CG^{i(\mathcal{K})}_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, she has a strategy to win $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ as well.

Proof. We prove the claim by induction on $m \in \mathbb{N}$. The case $m = 0$ follows immediately from that fact the the games CG'_m and CG^κ rely on the same winning condition.

Suppose that Duplicator has a strategy to win $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ for some $m > 0$. Let $X' \subseteq X$ be the set Spoiler chooses in $CG_m^{i(\mathcal{K})-1}(\pi_A, \bar{a}, \pi_B, \bar{b})$. Since $|X'| < i(\mathcal{K})$ and $i(0) = 1$, there must be a function $f\colon X \to K$ such that $f(x) = 0$ if, and only if, $x \in X \setminus X'$ and $|f^{-1}(k)| \leq i(k)$ for each $k \in \mathcal{K} \setminus \{0\}$. Let g be the function Duplicator responds to f in her winning strategy for $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. To win $CG_m^{i(\mathcal{K})-1}(\pi_A, \bar{a}, \pi_B, \bar{b})$, Duplicator answers $Y' = Y \setminus g^{-1}(0)$. Let $y \in Y'$ be the element Spoiler chooses afterwards. Duplicator answers with her response to y in her winning strategy for $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ which must be in X' by construction of f. As the resulting position is reachable in $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ if Duplicator plays according to her winning strategy, she wins remaining subgame of $CG_m^{i(\mathcal{K})-1}(\pi_A, \bar{a}, \pi_B, \bar{b})$ by induction.

Assume now that Duplicator wins $CG_m^{i(\mathcal{K})}(\pi_A, \bar{a}, \pi_B, \bar{b})$ and let $f\colon X \to K$ be the mapping Spoiler chooses in $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$. Let $\sim$ be the equivalence relation on $A \dot\cup B$ such that $x \sim y$ if, and only if, Duplicator wins each play of $CG_{m-1}^{i(\mathcal{K})}(\pi_X, \bar{x}, x, \pi_Y, \bar{y}, y)$ where $(\pi_X, \bar{x}), (\pi_Y, \bar{y}) \in \{(\pi_A, \bar{a}), (\pi_B, \bar{b})\}$. Let Z be an equivalence class with respect to $\sim$. It must hold that $|Z \cap A| = |Z \cap B|$ or both $|Z \cap A| \geq i(\mathcal{K})$ and $|Z \cap B| \geq i(\mathcal{K})$, because otherwise Spoiler would win $CG_m^{i(\mathcal{K})}(\pi_A, \bar{a}, \pi_B, \bar{b})$. Based on Z, we construct the mapping $g_Z\colon Z \cap Y \to K$ as follows. If $|Z \cap A| = |Z \cap B|$, let g_Z be an arbitrary mapping such that $|f^{-1}(k) \cap Z| = |g_Z^{-1}(k)|$ for each $k \in K$. Otherwise, let Z'_X be a maximal subset of $Z \cap X$ such that $|f^{-1}(k) \cap Z'_X| \leq i(k)$ for each k. By definition, it must hold that $|Z'_X| \leq i(\mathcal{K})$, hence there must be some $Z'_Y \subseteq Z \cap Y$ with $|Z'_X| = |Z'_Y|$. Since $|Z \cap X| \geq i(\mathcal{K})$, there must be some $\ell \in f(X \cap Z)$ such that $|f^{-1}(\ell)| \geq i(\ell)$. We choose g_Z as an arbitrary mapping such that $|f^{-1}(k) \cap Z'_X| = |g_Z^{-1}(k) \cap Z'_Y|$ for each $k \in K$ and $g_Z(y) = \ell$ for all $y \in Z \cap Y \setminus Z'_Y$. In $CG'_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, Duplicator can provide the combination g of all mappings g_Z. By construction of each g_Z, for each $y \in Y$, there is some $x \in X$ with $f(x) = g(y)$ and $x \sim y$. Hence, Duplicator wins the remaining subgame by induction. $\square$

There are simple counterexamples which illustrate that CG'_m is neither equivalent to $CG^{i(\mathcal{K})-1}$ nor to $CG^{i(\mathcal{K})}$. However, theorem 5.6 can be used to assess the completeness of CG'_m as a proof method for m-equivalence. As the finiteness of K and $i(k) < \omega$ for each $k \in K$ imply that $\mathcal{K}$ must be κ-idempotent where $\kappa = \max\{i(k)\colon k \in K\}$, m-equivalence is already ensured if Duplicator wins $CG_m^\kappa(\pi_A, \pi_B)$, as we proved in section 3.2. With $\kappa \leq i(\mathcal{K}) - 1$, the theorem above illustrates that incorporating the individual measures $i(k)$ into the game as described by CG'_m does not lead to a proof method for m-equivalence which is more general than CG^{κ}_m. Thus, we will now discuss a second approach to modifying the rules of the classical Ehrenfeucht-Fraïssé game.

5.2 Reduction to $\mathbb{B}$ via Homomorphisms

As several examples within chapter 3 illustrate, the fundamental property turns out to be central for proving m-equivalence of $\mathcal{K}$-interpretations. In particular, using the notion of separating sets of homomorphisms introduced in [GM21] enables us to reduce the problem of deciding whether to given $\mathcal{K}$-interpretations are m-equivalent to another semiring $\mathcal{L}$. Recall that a set H of homomorphisms from $\mathcal{K}$ to $\mathcal{L}$ is separating if for any two elements $k, k' \in \mathcal{K}$, there is some $h \in H$ such that $h(k) \neq h(k')$. Due to the fundamental property, m-equivalence of $\mathcal{K}$-interpretations corresponds to m-equivalence under any of these homomorphisms. In this manner, separating sets of homomorphisms allow reasoning in a simpler semiring, in particular, in the Boolean semiring. The fact that $\mathbb{B}$ corresponds to standard semantics gives rise to apply the standard Ehrenfeucht-Fraïssé game to prove or disprove m-equivalence of the resulting $\mathbb{B}$-interpretations. Hence, for the class of semirings $\mathcal{K}$ for which there is a separating set of homomorphisms to $\mathbb{B}$, we might be able to derive a *homomorphism game* capturing m-equivalence of $\mathcal{K}$-interpretations as follows.

(1) Spoiler chooses a homomorphism $h \in H$ from a separating set H of homomorphisms mapping $\mathcal{K}$ to $\mathbb{B}$.

(2) The players proceed with the game $G_m(h \circ \pi_A, h \circ \pi_B)$ played on the resulting $\mathbb{B}$-interpretations $h \circ \pi_A$ and $h \circ \pi_B$.

In order to prove the correctness of the game, it suffices to show that the game $G_m(h \circ \pi_A, h \circ \pi_B)$ indeed characterizes $h \circ \pi_A \equiv_m h \circ \pi_B$. Invoking theorem 3.5, we can immediately infer that a winning strategy of Duplicator in $G_m(h \circ \pi_A, h \circ \pi_B)$ assures $h \circ \pi_A \equiv_m h \circ \pi_B$, since $\mathbb{B}$ is fully idempotent. However, it not ensured that the resulting $\mathbb{B}$-interpretations $h \circ \pi_A$ and $h \circ \pi_B$ are model-defining, even if the $\mathcal{K}$-interpretations in question are assumed to be model-defining, which marks a crucial difference to classical semantics. Consider, for instance, $\mathcal{K}_2$-interpretations π_A^{13} and π_B^{13} where $\mathcal{K}_2 = (\{0, 1, 2\}, \max, \min, 0, 2)$ is a min-max semiring. The universes of π_A^{13} and π_B^{13} are given by $A := \{a_i : i \in \mathbb{N}\}$ and $B := \{b_i : i \in \mathbb{N}\} \cup \{b'_0\}$. Further, the semiring valuations are defined according to the following tables, which are supposed to indicate for $i > 0$ and $j \in \{1, 2\}$ that $\pi_A(R_j a_i) = \pi_B(R_j b_i) = 1$ if i is odd and $\pi_A(R_j a_i) = \pi_B(R_j b_i) = 2$ if i is even, while all negated τ-literals over A

and B are valuated with 0.

π_A^{13} :

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_0	2	1	0	0
a_1	1	1	0	0
a_2	2	2	0	0
a_3	1	1	0	0
a_4	2	2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

π_B^{13} :

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_0	1	1	0	0
b_0'	2	2	0	0
b_1	1	1	0	0
b_2	2	2	0	0
b_3	1	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We have already seen that the set $H := \{h_k \colon 0 < k \leq 2\}$ of homomorphisms $h_k \colon \mathcal{K}_2 \to \mathbb{B}$ with $h_k \colon \ell \mapsto 1$ if, and only if, $k \leq \ell$ is separating. For $k = 2$, we obtain the following $\mathbb{B}$-interpretations.

$h_2 \circ \pi_A^{13}$:

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_0	1	0	0	0
a_1	0	0	0	0
a_2	1	1	0	0
a_3	0	0	0	0
a_4	1	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$h_2 \circ \pi_B^{13}$:

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_0	0	0	0	0
b_0'	1	1	0	0
b_1	0	0	0	0
b_2	1	1	0	0
b_3	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Clearly, $h_2 \circ \pi_A^{13}$ and $h_2 \circ \pi_B^{13}$ are not model-defining, although π_A^{13} and π_B^{13} are. Since the element a_0 in $h_2 \circ \pi_A^{13}$ cannot be duplicated in $h_2 \circ \pi_B^{13}$, Spoiler wins the game $G_1(h_2 \circ \pi_A^{13}, h_2 \circ \pi_B^{13})$, so we would expect that $(h_2 \circ \pi_A^{13}) \not\equiv_1 (h_2 \circ \pi_B^{13})$. However, the $\mathbb{B}$-interpretations cannot be separated by $\exists x(R_1 x \wedge \neg R_2 x)$, as all negated τ-literals over A and B are valuated with 0. In fact, it can be shown that $h_2 \circ \pi_A^{13}$ and $h_2 \circ \pi_B^{13}$ are even elementarily equivalent, which illustrates that the classical Ehrenfeucht-Fraïssé does not capture m-equivalence or elementary equivalence of $\mathbb{B}$-interpretations which are not model-defining.

Proposition 5.7. The $\mathbb{B}$-interpretations $\pi_A^{14} := h_2 \circ \pi_A^{13}$ and $\pi_B^{14} := h_2 \circ \pi_B^{13}$ are elementarily equivalent.

Proof. We show that for each $i_1, \ldots, i_n \in \mathbb{N}$ and formula $\varphi(x_1, \ldots, x_n)$, it holds that

$$\pi_B^{14}[\![\varphi(\bar{b})]\!] \leq \pi_A^{14}[\![\varphi(\bar{a})]\!] \leq \pi_B^{14}[\![\varphi(\bar{b}')]\!],$$

where $\bar{a} = (a_{i_1}, \ldots, a_{i_n})$, $\bar{b} = (b_{i_1}, \ldots, b_{i_n})$ and $\bar{b}' \in B^n$ coincides with $\bar{b}$ up to occurrences of b_0 which are substituted by b_0'. The claim can be shown by induction on the structure of $\varphi(\bar{x})$ for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}$ simultaneously.

Case 1. If $\varphi(\bar{x})$ is an (in)equality, it holds that $\pi_B^{14}[\![\varphi(\bar{b})]\!] = \pi_A^{14}[\![\varphi(\bar{a})]\!] = \pi_B^{14}[\![\varphi(\bar{b}')]\!]$, as all (in)equalities are preserved when translating $\bar{a}$ into $\bar{b}$ or $\bar{b}'$, which rely on the same indices $i_1, \ldots, i_n$.

Case 2. For τ-literals $\varphi(\bar{x})$, the claim follows immediately from the definition of π_A^{14} and π_B^{14}.

Case 3. Let $\varphi(\bar{x}) = \psi(\bar{x}) \vee \vartheta(\bar{x})$ and suppose that $\pi_A^{14}[\![\varphi(\bar{a})]\!] = 0$. It suffices to show that $\pi_B^{14}[\![\varphi(\bar{b})]\!] = 0$ in this case, as $\pi_A^{14}[\![\varphi(\bar{a})]\!] \leq \pi_B^{14}[\![\varphi(\bar{b}')]\!]$ is clearly satisfied. We have that $\pi_A^{14}[\![\psi(\bar{a})]\!] = \pi_A^{14}[\![\vartheta(\bar{a})]\!] = 0$, implying $\pi_B^{14}[\![\psi(\bar{b})]\!] = \pi_B^{14}[\![\vartheta(\bar{b})]\!] = 0$ by induction hypothesis. Hence, $\pi_B^{14}[\![\varphi(\bar{b})]\!] = 0$ and we obtain $\pi_B^{14}[\![\varphi(\bar{b})]\!] \leq \pi_A^{14}[\![\varphi(\bar{a})]\!]$. Otherwise, it must hold that $\pi_A^{14}[\![\varphi(\bar{a})]\!] = 1$, yielding $\pi_A^{14}[\![\psi(\bar{a})]\!] = 1$ or $\pi_A^{14}[\![\vartheta(\bar{a})]\!] = 1$. Applying the induction hypothesis yields $\pi_B^{14}[\![\psi(\bar{b}')]\!] = 1$ or $\pi_B^{14}[\![\vartheta(\bar{b}')]\!] = 1$. Hence, $\pi_B^{14}[\![\varphi(\bar{b}')]\!] = 1$ and we obtain $\pi_A^{14}[\![\varphi(\bar{a})]\!] \leq \pi_B^{14}[\![\varphi(\bar{b}')]\!]$, while $\pi_B^{14}[\![\varphi(\bar{b})]\!] \leq \pi_A^{14}[\![\varphi(\bar{a})]\!]$ follows immediately from $\pi_A^{14}[\![\varphi(\bar{a})]\!] = 1$.

Case 4. Suppose that $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$ and $\pi_A^{14}[\![\varphi(\bar{a})]\!] = 0$. Then, it must hold that $\pi_A^{14}[\![\psi(\bar{a}, a)]\!] = 0$ for all $a \in A$, which implies $\pi_B^{14}[\![\psi(\bar{b}, b)]\!] = 0$ for all $b \in B \backslash \{b_0'\}$ by induction hypothesis. Fix some $b \in B$ which is not contained in $\bar{b}$ such that $\pi_B^{14}(R_1 b) = \pi_B^{14}(R_2 b) = 0$. It holds that $(\pi_B^{14}, \bar{b}, b_0') \cong (\pi_B^{14}, \bar{b}, b)$, so applying the isomorphism lemma yields $\pi_B^{14}[\![\psi(\bar{b}, b_0')]\!] = 0$. We overall obtain $\pi_B^{14}[\![\varphi(\bar{b})]\!] = 0$, so $\pi_B^{14}[\![\varphi(\bar{b})]\!] \leq \pi_A^{14}[\![\varphi(\bar{a})]\!]$. In case $\pi_A^{14}[\![\varphi(\bar{a})]\!] = 1$, there must be some $a_i \in A$ such that $\pi_A^{14}[\![\psi(\bar{a}, a_i)]\!] = 1$. It follows from the induction hypothesis that $\pi_B^{14}[\![\psi(\bar{b}', b_i)]\!] = 1$ if $i > 0$ and $\pi_B^{14}[\![\psi(\bar{b}', b_0')]\!] = 1$ in the case $i = 0$. Thus, it holds that $\pi_B^{14}[\![\varphi(\bar{b}')]\!] = 1$, which yields $\pi_A^{14}[\![\varphi(\bar{a})]\!] \leq \pi_B^{14}[\![\varphi(\bar{b}')]\!]$.

We omit the cases $\varphi(\bar{x}) = \psi(\bar{x}) \wedge \vartheta(\bar{x})$ and $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$, as they are analogous to case 3 and 4. In particular, the inequality implies that $\pi_B^{14}[\![\varphi]\!] \leq \pi_A^{14}[\![\varphi]\!] \leq \pi_B^{14}[\![\varphi]\!]$ for all sentences φ, hence we obtain $\pi_A^{14} \equiv \pi_B^{14}$. □

The problems arising when applying the game G_m to $\mathbb{B}$-interpretations which are not model-defining are not limited to infinite universes. It is easy to construct finite counterexamples based on the elementary equivalence of π_A^{14} and π_B^{14} by considering certain subinterpretations. For each $m \in \mathbb{N}_{>0}$, let $\pi_A^{14,m}$ and $\pi_B^{14,m}$ be the subinterpretations of π_A^{14} and π_B^{14} induced by the sets $\{a_i \colon 0 \leq i \leq 2m\}$ and $\{b_i \colon 1 \leq i \leq 2m\}$. Clearly, Duplicator wins both $G_m(\pi_A^{14,m}, \pi_A^{14})$ and $G_m(\pi_B^{14,m}, \pi_B^{14})$. Using $\pi_A^{14} \equiv \pi_B^{14}$, so in particular $\pi_A^{14} \equiv_m \pi_B^{14}$, we can conclude that $\pi_A^{14,m} \equiv_m \pi_B^{14,m}$ by transitivity of m-equivalence. Still, Spoiler wins the game $G_1(\pi_A^{14,m}, \pi_B^{14,m})$ by drawing a_0 from A.

$$
\begin{array}{ccc}
\pi_A^{14} & \equiv & \pi_B^{14} \\
\equiv_1 & & \equiv_1 \\
\pi_A^{14,1}: & \equiv_1 & \pi_B^{14,1}:
\end{array}
$$

A	R_1	R_2	$\neg R_1$	$\neg R_2$
a_0	1	0	0	0
a_1	0	0	0	0
a_2	1	1	0	0

B	R_1	R_2	$\neg R_1$	$\neg R_2$
b_1	0	0	0	0
b_2	1	1	0	0

Theorem 5.8. There are $\mathbb{B}$-interpretations π_A and π_B (which are not model-defining) such that Spoiler wins $G_1(\pi_A, \pi_B)$ and $\pi_A \equiv \pi_B$. Further, for each $m \in \mathbb{N}_{>0}$, there are finite m-equivalent $\mathbb{B}$-interpretations π_A^m and π_B^m such that Spoiler wins $G_1(\pi_A^m, \pi_B^m)$.

Returning to the initial $\mathcal{K}_2$-interpretations π_A^{13} and π_B^{13}, we can conclude that $\pi_A^{13} \equiv \pi_B^{13}$, as we just proved that $h_2 \circ \pi_B^{13} \equiv h_2 \circ \pi_B^{13}$ and $h_1 \circ \pi_B^{13} \equiv h_1 \circ \pi_B^{13}$ follows from $h_1 \circ \pi_B^{13} \cong h_1 \circ \pi_B^{13}$. Yet, Spoiler wins the homomorphism game on π_A^{13} and π_B^{13} with respect to the separating homomorphism set $\{h_1, h_2\}$ by choosing h_1 and $a_0 \in A$ afterwards. The analogous with regard to m-equivalence can be observed for the $\mathcal{K}_2$-interpretations $\pi_A^{13,m}$ and $\pi_B^{13,m}$ which arise from π_A^{13} and π_B^{13} in the same way as $\pi_A^{14,m}$ and $\pi_B^{14,m}$ are derived from π_A^{14} and π_B^{14}. Hence, we can conclude that the homomorphism game, as described before, neither characterizes m-equivalence, nor elementary equivalence of $\mathcal{K}_2$-interpretations.

In order to construct an appropriate game based on a separating set of homomorphisms to $\mathbb{B}$, we need to adjust the classical Ehrenfeucht-Fraïssé game such that it can be applied to arbitrary $\mathbb{B}$-interpretations which are not necessarily model-defining. Note that omitting the requirement that the considered $\mathbb{B}$-interpretations are model-defining affects the expressive power of FO, since the only negation that can be expressed refers to inequalities of elements. As opposed to equalities, the valuation $\pi_A(\neg R\bar{a})$ does not contain any information about $\pi_A(R\bar{a})$, hence $\neg R$ can be considered as an additional, independent relation symbol. In order to account for the restricted negation, we adjust the game G_m as follows.

Definition 5.9. Let π_A and π_B be $\mathbb{B}$-interpretations. In the game $G'_m(\pi_A, \pi_B)$ Spoiler first chooses one of the $\mathbb{B}$-interpretations $\pi \in \{\pi_A, \pi_B\}$. Accordingly, the remaining subgame is denoted as $G'_m(\underline{\pi_A}, \pi_B)$ if $\pi = \pi_A$ and as $G'_m(\pi_A, \underline{\pi_B})$ in case $\pi = \pi_B$. In each of m turns in total, Spoiler chooses $a \in A$ or $b \in B$ and Duplicator has to respond with some element in the other structure. Hence, each play ends in some position $(\pi, \bar{a}, \bar{b})$ where $\bar{a} \in A^m$ and $\bar{b} \in B^m$. Duplicator wins the play if, and only if, for each $L(\bar{x}) \in \mathrm{Lit}_m(\tau)$

$$\pi_A(L(\bar{a})) \leq \pi_B(L(\bar{b})) \text{ if } \pi = \pi_A \text{ and}$$
$$\pi_B(L(\bar{b})) \leq \pi_A(L(\bar{a})) \text{ if } \pi = \pi_B.$$

In accordance with the original game G_m, we denote the subgame of G'_m after the i-th step as $G'_{m-i}(\underline{\pi_A}, a_1, \dots, a_i, \pi_B, b_1, \dots, b_i)$ or $G'_{m-i}(\pi_A, a_1, \dots, a_i, \underline{\pi_B}, b_1, \dots, b_i)$, depending on the initial choice of the $\mathcal{K}$-interpretation. Despite the fact that it is easier for Duplicator to win $G'_m(\pi_A, \pi_B)$ instead of the classical game $G_m(\pi_A, \pi_B)$, it can be shown that a winning strategy in G'_m still ensures m-equivalence.

Proposition 5.10. Let π_A, π_B be $\mathbb{B}$-interpretations and $\bar{a} \in A^n$, $\bar{b} \in B^n$ be tuples of elements. If there is a formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\mathrm{qr}(\varphi(\bar{x})) = m$ such that $\pi_A[\![\varphi(\bar{a})]\!] > \pi_B[\![\varphi(\bar{b})]\!]$, then Spoiler wins $G'_m(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$.

Proof. We prove the claim by induction on the structure of $\varphi(\bar{x})$.

Case 1. If $\varphi(\bar{x})$ is a literal, then $\pi_A[\![\varphi(\bar{a})]\!] > \pi_B[\![\varphi(\bar{b})]\!]$ violates the winning condition. Hence, Spoiler wins $G'_0(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$.

Case 2. If $\varphi(\bar{x}) = \psi(\bar{x}) \vee \vartheta(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) = m$, then $\pi_A[\![\varphi(\bar{a})]\!] > \pi_B[\![\varphi(\bar{b})]\!]$ implies that $\pi_A[\![\psi(\bar{a})]\!] = 1$ or $\pi_A[\![\vartheta(\bar{a})]\!] = 1$, whereas $\pi_B[\![\psi(\bar{b})]\!] = \pi_B[\![\vartheta(\bar{b})]\!] = 0$. Hence, it must hold that $\pi_A[\![\psi(\bar{a})]\!] > \pi_B[\![\psi(\bar{b})]\!]$ or $\pi_A[\![\vartheta(\bar{a})]\!] > \pi_B[\![\vartheta(\bar{b})]\!]$. Applying the induction hypothesis yields that Spoiler wins the game $G'_m(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$.

Case 3. For $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$ with $\mathrm{qr}(\varphi(\bar{x})) = m$ it follows from the assumption $\pi_A[\![\varphi(\bar{a})]\!] > \pi_B[\![\varphi(\bar{b})]\!]$ that $\pi_A[\![\psi(\bar{a}, a)]\!] = 1$ for some $a \in A$. In the game $G'_m(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$, Spoiler can pick this element $a \in A$. For any possible answer $b \in B$, it must hold that $\pi_B[\![\psi(\bar{b}, b)]\!] = 0$, as $\pi_B[\![\varphi(\bar{b})]\!] = 0$ by assumption. Hence, with the induction hypothesis we obtain that Spoiler wins the remaining subgame $G'_{m-1}(\underline{\pi_A}, \bar{a}, a, \pi_B, \bar{b}, b)$.

The cases $\varphi(\bar{x}) = \psi(\bar{x}) \wedge \vartheta(\bar{x})$ as well as $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$ are omitted, as the reasoning is analogous to cases 2 and 3. $\square$

To prove the converse implication, we inductively construct characteristic formulae $\chi^m_{\pi_A, \bar{a}}(x_1, \dots, x_n)$, analogously to the proof of the classical Ehrenfeucht-Fraïssé theorem. As opposed to the construction for classical structures, we do not incorporate literals of the form $\neg R\bar{x}$ into $\chi^0_{\pi_A, \bar{a}}(x_1, \dots, x_n)$ if $\pi_A(R\bar{a}) = 0$ but only take into account the τ-literals which are valuated with 1 by $(\pi_A, \bar{a})$. Recall that $\varphi^=_{\bar{a}}(\bar{x})$ describes the equalities and inequalities of the elements in $\bar{a}$.

$$\chi^0_{\pi_A, \bar{a}}(x_1, \dots, x_n) := \varphi^=_{\bar{a}}(\bar{x}) \wedge \bigwedge \{L(\bar{x}) \in \mathrm{Lit}_n(\tau) | \pi_A(L(\bar{a})) = 1\}$$
$$\chi^{m+1}_{\pi_A, \bar{a}}(x_1, \dots, x_n) := \bigwedge_{a \in A} \exists x \; \chi^m_{\pi_A, \bar{a}, a}(\bar{x}, x) \wedge \forall x \bigvee_{a \in A} \chi^m_{\pi_A, \bar{a}, a}(\bar{x}, x)$$

Theorem 5.11. For any two $\mathbb{B}$-interpretations π_A and π_B with elements $\bar{a} \in A^n$ and B^n and any $m \in \mathbb{N}$, the following are equivalent:

(1) Duplicator wins $G'_m(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$

(2) $\pi_B[\![\chi^m_{\pi_A,\bar{a}}(\bar{b})]\!] = 1$

(3) $\pi_A[\![\varphi(\bar{a})]\!] \leq \pi_B[\![\varphi(\bar{b})]\!]$ for all $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\mathrm{qr}(\varphi(\bar{x})) \leq m$

Proof. We have already proven implication (1) $\Rightarrow$ (3) by contraposition. It follows from the definition of $\chi^m_{\pi_A,\bar{a}}(\bar{x})$ that $\pi_A[\![\chi^m_{\pi_A,\bar{a}}(\bar{a})]\!] = 1$, hence we can conclude that (3) $\Rightarrow$ (2) holds as well. Consequently, it remains to show (2) $\Rightarrow$ (1), which we prove by induction on $m \in \mathbb{N}$. If $\pi_B[\![\chi^0_{\pi_A,\bar{a}}(b_1, \ldots, b_n)]\!] = 1$, then the winning condition for $\pi = \pi_A$ must be fulfilled, so Duplicator wins $G'_0(\underline{\pi_A}, \bar{a}, \pi_B, \bar{b})$. If $\pi_B[\![\chi^{m+1}_{\pi_A,\bar{a}}(b_1, \ldots, b_n)]\!] = 1$, then for each $a \in A$, there must be some $b \in B$ such that $\pi_B[\![\chi^m_{\pi_A,\bar{a},a}(b_1, \ldots, b_n, b)]\!] = 1$ and for each $b \in B$, there must be some $a \in A$ such that $\pi_B[\![\chi^m_{\pi_A,\bar{a},a}(b_1, \ldots, b_n, b)]\!] = 1$. It follows by induction that in both cases Duplicator wins the remaining subgame $G'_m(\underline{\pi_A}, \bar{a}, a, \pi_B, \bar{b}, b)$. □

Clearly, the above theorem implies that m-equivalence of any two $\mathbb{B}$-interpretations π_A and π_B is characterized by $G'_m(\pi_A, \pi_B)$, regardless of whether π_A and π_B are model-defining. With the classical Ehrenfeucht-Fraïssé game on $\mathbb{B}$-interpretations being adapted, we return to the game we aim to derive based on a separating set of homomorphisms H into $\mathbb{B}$. Based on the idea that Spoiler first chooses a homomorphism and the game G'_m is played on the resulting $\mathbb{B}$-interpretations afterwards, we construct the *homomorphism game* HG_m as follows.

Definition 5.12. Let π_A and π_B be $\mathcal{K}$-interpretations and H be a separating set of homomorphisms from $\mathcal{K}$ to $\mathbb{B}$. In the game $HG_m(H, \pi_A, \pi_B)$, Spoiler first chooses $\pi \in \{\pi_A, \pi_B\}$ and a homomorphism $h \in H$. Thereupon, Spoiler chooses $a \in A$ or $b \in B$ and Duplicator has to respond with some element in the other structure, which is repeated m times. Duplicator wins a play resulting in the position $(\pi, h, a_1, \ldots, a_m, b_1, \ldots, b_m)$ if, and only if, for each $L(\bar{x}) \in \mathrm{Lit}_n(\tau)$,

$$h(\pi_A(L(\bar{a}))) \leq h(\pi_B(L(\bar{b}))) \text{ if } \pi = \pi_A \text{ and}$$
$$h(\pi_B(L(\bar{b}))) \leq h(\pi_A(L(\bar{a}))) \text{ if } \pi = \pi_B.$$

As a direct consequence of theorem 5.11 and the fact that Duplicator wins the game $HG_m(H, \pi_A, \pi_B)$ if, and only if, she wins $G'_m(h \circ \pi_A, h \circ \pi_B)$ for all $h \in H$, the correctness of the homomorphism game can be derived as follows.

Theorem 5.13. Let $\mathcal{K}$ be a semiring with a separating set H of homomorphisms $h \colon \mathcal{K} \to \mathbb{B}$. Given any two $\mathcal{K}$-interpretations π_A and π_B with elements $\bar{a} \in A^n$ and B^n and any $m \in \mathbb{N}$, the following are equivalent:

(1) Duplicator wins $HG_m(H, \pi_A, \bar{a}, \pi_B, \bar{b})$

(2) $h(\pi_B[\![\chi^m_{h\circ\pi_A,\bar{a}}(\bar{b})]\!]) = h(\pi_A[\![\chi^m_{h\circ\pi_B,\bar{b}}(\bar{a})]\!]) = 1$ for each $h \in H$

(3) $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$

Having established a game which captures m-equivalence in any semiring for which there is a separating set of homomorphisms mapping to $\mathbb{B}$, the question arises to which semirings this condition applies. So far, we have only seen how a separating homomorphism set with respect to $\mathbb{B}$ can be constructed for min-max semirings. Further, the formulation of the game HG_m is rather abstract, as Spoiler draws a homomorphism we do not know much about yet. If we were able to construct the separating homomorphism set systematically, it could directly be embedded into the game rules.

Certainly, there are semirings $\mathcal{K}$ containing distinct elements which cannot be separated by a homomorphism mapping $\mathcal{K}$ to $\mathbb{B}$. For instance, for any homomorphism $h\colon \mathcal{K} \to \mathbb{B}$ and $k, \ell \in \mathcal{K}$, it holds that

$$h(k \cdot^{\mathcal{K}} k) = h(k) \wedge h(k) = h(k) \text{ and}$$
$$h(k +^{\mathcal{K}} k\ell) = h(k) \vee (h(k) \wedge k(\ell)) = h(k),$$

following from multiplicative idempotence and absorption in $\mathbb{B}$. Hence, we can conclude that each semiring for which there is a separating set of homomorphisms to $\mathbb{B}$ must be multiplicatively idempotent and absorptive. It can easily be verified that the class of multiplicatively idempotent and absorptive semirings corresponds to *distributive lattices* with least element 0 and greatest element 1.

In the following, we present an explicit construction of a separating set of homomorphisms to the Boolean semiring, which is motivated by *Birkhoff's representation theorem* [Bir37] and applies to finite distributive lattices. By incorporating this set directly into the rules of HG_m, we obtain a more intuitive formulation of the game.

5.2.1 Application to Finite Distributive Lattices

For the remainder of this section, let $\mathcal{K} = (K, +, \cdot, 0, 1)$ be a finite distributive lattice and suppose that the associated infinitary operations are also fully idempotent. The homomorphisms $h_k\colon \mathcal{K} \to \mathbb{B}$ we construct depend on a certain semiring element $k \in \mathcal{K}$. In order to ensure the compatibility of h_k with addition in $\mathcal{K}$, the element k must be indecomposable with respect to addition in the following sense.

Definition 5.14. A non-zero element $k \in \mathcal{K}$ is said to be $+$-*indecomposable* if for all $\ell_1, \ell_2 \in \mathcal{K}$ with $\ell_1 \neq k$ and $\ell_2 \neq k$ it holds that $\ell_1 + \ell_2 \neq k$. We denote the set of non-zero $+$-indecomposable of elements in $\mathcal{K}$ as $idc(\mathcal{K})$.

In a min-max semiring, for instance, every non-zero element is $+$-indecomposable. By contrast, the $+$-indecomposable elements in PosBool$[X]$ correspond to the monomials.

Proposition 5.15. For each $k \in idc(\mathcal{K})$, the mapping $h_k \colon \mathcal{K} \to \mathbb{B}$ defined by

$$h_k(\ell) = \begin{cases} 1, & k + \ell = \ell \\ 0, & \text{otherwise} \end{cases} .$$

is a homomorphism from $\mathcal{K}$ into $\mathbb{B}$.

Proof. Let $k \in idc(\mathcal{K})$ be non-zero and $+$-indecomposable.

(1) Since $k + 0 = k \neq 0$, it holds that $h_k(0) = 0$. Further, we have that $k + 1 = 1 \cdot k + 1 = 1$ due to absorption, hence $h_k(1) = 1$.

(2) In order to prove that $h_k(\ell_1 + \ell_2) = h_k(\ell_1) + h_k(\ell_2)$ for all $\ell_1, \ell_2 \in \mathcal{K}$, it remains to show that $k + (\ell_1 + \ell_2) = \ell_1 + \ell_2$ is equivalent to $k + \ell_1 = \ell_1$ or $k + \ell_2 = \ell_2$. If $k + (\ell_1 + \ell_2) = \ell_1 + \ell_2$, then with absorption and distributivity $k\ell_1 + k\ell_2 = k(\ell_1 + \ell_2) = k(k + \ell_1 + \ell_2) = k + k(\ell_1 + \ell_2) = k$. Since k is $+$-indecomposable by assumption, this implies $k\ell_1 = k$ or $k\ell_2 = k$. Suppose w.l.o.g. that $k\ell_1 = k$ which yields $\ell_1 = \ell_1 + k\ell_1 = \ell_1 + k$. For the converse implication, assume that $k+\ell_1 = \ell_1$ or $k+\ell_2 = \ell_2$. Clearly, both implications immediately yield $k + (\ell_1 + \ell_2) = \ell_1 + \ell_2$.

(3) To derive $h_k(\ell_1 \cdot \ell_2) = h_k(\ell_1) \cdot h_k(\ell_2)$, it has to be shown that $k + \ell_1\ell_2 = \ell_1\ell_2$ is equivalent to $k + \ell_1 = \ell_1$ and $k + \ell_2 = \ell_2$. If $k + \ell_1\ell_2 = \ell_1\ell_2$, we can infer that $k + \ell_1 = k + (\ell_1 + \ell_1\ell_2) = (k + \ell_1\ell_2) + \ell_1 = \ell_1\ell_2 + \ell_1 = \ell_1$ and the analogous for ℓ_2. For the converse, suppose that $k + \ell_1 = \ell_1$ and $k + \ell_2 = \ell_2$. Then, $\ell_1\ell_2 = (k + \ell_1)(k + \ell_2) = k + (\ell_1 \cdot \ell_2)$ follows by distributivity.

(4) Pertaining to the compatibility of h_k with the infinitary operations in $\mathcal{K}$, note that for each $(\ell_i)_{i \in I}$ we have that $\sum_{i \in I} \ell_i = \sum_{\ell \in L} \ell$ and $\prod_{i \in I} \ell_i = \prod_{\ell \in L} \ell$ where $L := \{\ell_i \colon i \in I\}$ by full idempotence of the infinitary operations. Since $\mathcal{K}$ is finite, L must be finite as well. Consequently, the compatibility of h_k with the infinitary operations in $\mathcal{K}$ follows readily from (2) and (3) by induction. $\square$

Although we only consider the mappings h_k for $+$-indecomposable k to ensure that h_k is a homomorphism, any two elements in $\mathcal{K}$ can be separated by some h_k.

Proposition 5.16. The set $H_{idc} := \{h_k \colon k \in idc(\mathcal{K})\}$ is a separating set of homomorphisms from $\mathcal{K}$ to $\mathbb{B}$.

Proof. For $\ell \in \mathcal{K}$ let $S_\ell = \{k \in \mathcal{K} \in idc(\mathcal{K}) \colon k + \ell = \ell\}$. Due to idempotence, we have that $\ell + \sum_{k \in S_\ell} k = \ell$. Full idempotence of $\mathcal{K}$ ensures that $\mathcal{K}$ is naturally ordered, so the natural order is in particular antisymmetric. As $\mathcal{K}$ is finite, this implies that the natural order is well-founded, hence there must be a tuple $\ell_1, \dots, \ell_n \in idc(\mathcal{K})$ with $\ell_1 + \dots + \ell_n = \ell$. With idempotence, this implies $\ell + \ell_i = \ell$, which yields $\ell_i \in S_\ell$ for each $1 \leq i \leq n$. Hence, we have that $\ell + \sum_{k \in S_\ell} k = \sum_{1 \leq i \leq n} \ell_i + \sum_{k \in S_\ell} k = \sum_{k \in S_\ell} k$. Overall, we obtain $\ell = \ell + \sum_{k \in S_\ell} k = \sum_{k \in S_\ell} k$.

Let $\ell_1, \ell_2 \in \mathcal{K}$ with $\ell_1 \neq \ell_2$. Since $\ell_1 = \sum_{k \in S_{\ell_1}} k$ and $\ell_2 = \sum_{k \in S_{\ell_2}} k$, it must hold that $S_{\ell_1} \neq S_{\ell_2}$. Let k be a witness for the inequality and assume w.l.o.g that $k \in S_{\ell_1}$. By definition of S_{ℓ_1}, it holds that $k + \ell_1 = \ell_1$, hence $h_k(\ell_1) = 1$. By contrast, $k \notin S_{\ell_2}$ yields $k + \ell_2 \neq \ell_2$ ans thus $h_k(\ell_2) = 0$. $\square$

As we derived an explicit construction of a separating set of homomorphisms to $\mathbb{B}$ which applies to any finite distributive lattice, we can reformulate the homomorphism game as $HG_m^{fin}(\pi_A, \pi_B)$ corresponding to $HG_m(H_{idc}, \pi_A, \pi_B)$ for finite distributive lattices as follows.

Definition 5.17. At the beginning of each play in $HG_m^{fin}(\pi_A, \pi_B)$, Spoiler chooses $\pi \in \{\pi_A, \pi_B\}$ and some $k \in idc(\mathcal{K})$. In the i-th of m rounds in total, Spoiler chooses some $a_i \in A$ or $b_i \in B$ and Duplicator has to respond with an element a_i or b_i in the other structure. The pair (a_i, b_i) is added to the current position such that at the end of each play some position $(\pi, k, a_1, \dots, a_m, b_1, \dots, b_m)$ is reached. The corresponding play is won by Duplicator, if and only if, for each $L(\bar{x}) \in \mathrm{Lit}_m(\tau)$,

$$\pi_A(L(\bar{a})) + k = \pi_A(L(\bar{a})) \text{ implies } \pi_B(L(\bar{b})) + k = \pi_B(L(\bar{b})) \text{ if } \pi = \pi_A \text{ and}$$
$$\pi_B(L(\bar{b})) + k = \pi_B(L(\bar{b})) \text{ implies } \pi_A(L(\bar{a})) + k = \pi_A(L(\bar{a})) \text{ if } \pi = \pi_B.$$

The direct construction of the separating homomorphism set also allows an explicit formulation of the characteristic formulae $\chi^{m,k}_{\pi_A,\bar{a}}(\bar{x})$ for each $k \in idc(\mathcal{K})$ corresponding to the $\mathbb{B}$-interpretations $h_k \circ \pi_A$.

$$\chi^{0,k}_{\pi_A,\bar{a}}(x_1, \dots, x_n) := \varphi^{=}_{\bar{a}}(\bar{x}) \wedge \bigwedge \{L(\bar{x}) \in \mathrm{Lit}_n(\tau) | \pi_A(L(\bar{a})) + k = \pi_A(L(\bar{a}))\}$$
$$\chi^{m+1,k}_{\pi_A,\bar{a}}(x_1, \dots, x_n) := \bigwedge_{a \in A} \exists x\, \chi^{m,k}_{\pi_A,\bar{a},a}(\bar{x}, x) \wedge \forall x \bigvee_{a \in A} \chi^{m,k}_{\pi_A,\bar{a},a}(\bar{x}, x)$$

In terms of the set $H_{idc} = \{h_k \colon k \in idc(\mathcal{K})\}$, the correctness of the game HG_m^{fin} for finite distributive lattices can be stated as follows.

Theorem 5.18. Let π_A, π_B be $\mathcal{K}$-interpretations with elements $\bar{a} \in A^n$ and $\bar{b} \in B^n$ where $\mathcal{K}$ is a finite distributive lattice with idempotent infinitary operations. For each $m \in \mathbb{N}$, the following are equivalent.

(1) Duplicator wins $HG_m^{fin}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

(2) For each $k \in idc(\mathcal{K})$, it holds that $\pi_B[\![\chi^{m,k}_{\pi_A,\bar{a}}(\bar{b})]\!] + k = \pi_B[\![\chi^{m,k}_{\pi_A,\bar{a}}(\bar{b})]\!]$ and $\pi_A[\![\chi^{m,k}_{\pi_B,\bar{b}}(\bar{a})]\!] + k = \pi_A[\![\chi^{m,k}_{\pi_B,\bar{b}}(\bar{a})]\!]$.

(3) $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$

Note that for min-max semirings, the constructed separating homomorphism set coincides with the homomorphisms we considered previously, since $k + \ell = \max(k, \ell) = \ell$ is equivalent to $k \leq \ell$ in a min-max semiring. For PosBool$[X]$, the set $\{h_k : k \in idc(\text{PosBool}[X])\}$ corresponds to the homomorphisms induced by variable assignments $X \to \{0, 1\}$. This is because every $m \in idc(\text{PosBool}[X])$ must be a monomial and $m + p = p$ is equivalent to m being absorbed by some monomial in p. Hence, the variable assignment which assigns 1 to all variables in m and 0 to all remaining variables satisfies exactly the polynomials such that $m + p = p$.

5.2.2 Transferability to Infinite Distributive Lattices

In the case of min-max semirings, the construction of the separating homomorphism set also applies to infinite semirings. However, it can be shown that the constructed set H_{idc} does not suffice to separate infinite distributive lattices in general. As an example, consider the algebraic structure $\mathcal{K} = (\mathbb{N}, +^{\mathcal{K}}, \cdot^{\mathcal{K}}, 0, 1)$ with $a +^{\mathcal{K}} b = \gcd(a, b)$ if $a \neq 0$ or $b \neq 0$, while $0 +^{\mathcal{K}} 0 = 0$ and $a \cdot^{\mathcal{K}} b = \text{lcm}(a, b)$ for $a, b \in \mathbb{N}$. It is straightforward to verify that $\mathcal{K}$ is a distributive lattice. For each $a \in \mathbb{N}$, it holds that $\gcd(2a, 3a) = a$, so for $a \neq 0$ there are distinct b and c such that $a = b +^{\mathcal{K}} c$. By contrast, $\gcd(a, b) \neq 0$ for all $a, b \in \mathbb{N} \setminus \{0\}$, hence $idc(\mathcal{K}) = \{0\}$. The function h_0 maps each $a \in \mathbb{N} \setminus \{0\}$ to 1, because $\gcd(a, 0) = a$ for all $a \in \mathbb{N} \setminus \{0\}$. Hence, h_0 does not separate 1 and 2, for instance, and the set $\{h_0\}$ is not a separating set of homomorphisms.

Nevertheless, a separating set of homomorphisms to $\mathbb{B}$ can also be constructed in the infinite case, resulting in a slightly more involved formulation of the homomorphism game. For the remainder of this section, let $\mathcal{K}$ be a distributive lattice with infinitary summation defined by the supremum of the finite subsums and, analogously, infinitary multiplication by the infimum of the finite subproducts. Instead of the +-indecomposable elements which are sufficient to separate finite distributive lattices, the construction for infinite distributive lattices relies on the *prime ideals* in $\mathcal{K}$.

Definition 5.19. Let $\mathcal{K}$ be a distributive lattice. A non-empty proper subset P of K is said to be a *prime ideal* if

(1) $k \in P$ and $\ell \in P$ imply $k + \ell \in P$,

(2) $k \in P$ and $\ell \in K$ imply $k \cdot \ell \in P$ and

(3) $k \cdot \ell \in P$ implies $k \in P$ or $\ell \in P$.

We denote the set of prime ideals in $\mathcal{K}$ by $I_p(\mathcal{K})$.

In order to derive a separating set of homomorphisms from the prime ideals in $\mathcal{K}$, it remains to show that there are sufficiently many prime ideals such that any two elements in $\mathcal{K}$ can be separated. Stone, in [Sto38], proved that for each pair of distinct elements $k, \ell \in \mathcal{K}$, there is a prime ideal which contains one of k and ℓ but not both. While this observation is crucial to *Stone's representation theorem for distributive lattices*, we will use it to construct a set of homomorphisms from $\mathcal{K}$ to $\mathbb{B}$ and prove that it is separating.

Lemma 5.20 ([Sto38])**.** The mapping $f\colon \mathcal{K} \to \mathcal{P}(I_p(\mathcal{K}))$ defined according to $f\colon k \mapsto \{P \in I_p(\mathcal{K})\colon k \notin P\}$ is injective and it holds for each $k, \ell \in K$ that

(1) $f(k + \ell) = f(k) \cup f(\ell)$ and

(2) $f(k \cdot \ell) = f(k) \cap f(\ell)$.

Proposition 5.21. The set $H_p := \{h_P\colon P \in I_P(\mathcal{K})\}$ of mappings $h_P\colon \mathcal{K} \to \mathbb{B}$ with $h_P\colon k \mapsto 0$ if, and only if, $k \in P$ is a separating set of homomorphisms.

Proof. Let $P \in I_P(\mathcal{K})$. We first verify that h_P is a homomorphism from $\mathcal{K}$ to $\mathbb{B}$.

By definition, it holds that $P \neq \varnothing$ which, with property (2) of prime ideals, implies that $0 \in P$. Hence, it holds that $h_P(0) = 0$. If $1 \in P$, we would have that $P = K$ due to property (2) of prime ideals, which yields a contradiction. Hence, it must hold that $1 \notin P$, so $h_P(1) = 1$, which proves the preservation of the neutral elements by h_P.

In order to verify the compatibility of h_P with addition and multiplication, it remains to show for each $k, \ell \in K$ that $k + \ell \notin P$ if, and only if, $k \notin P$ or $\ell \notin P$ and that $k \cdot \ell \notin P$ is equivalent to $k \notin P$ and $\ell \notin P$. Both equivalences follow immediately from property (1) and (2) of lemma 5.20.

Property (1) in 5.20 also implies that $k \leq \ell$ if, and only if, $f(k) \subseteq f(\ell)$ where $\leq$ refers to the natural order $(*)$. Hence, we can conclude that

$$f\Big(\sum_{i\in I} k_i\Big) = f\Big(\sup_{\substack{I'\subseteq I\\ \text{finite}}} \sum_{i\in I'} k_i\Big) \stackrel{(*)}{=} \sup_{\substack{I'\subseteq I\\ \text{finite}}} f\Big(\sum_{i\in I'} k_i\Big) \stackrel{(1)}{=} \sup_{\substack{I'\subseteq I\\ \text{finite}}} \bigcup_{i\in I'} f(k_i) = \bigcup_{i\in I} f(k_i)$$

for each family $(k_i)_{i\in I}$ of elements in $\mathcal{K}$. The equality implies that $\sum_{i\in I} k_i \notin P$ if, and only if, $k_i \notin P$ for some $i \in I$, i.e., h_P respects infinitary summation in $\mathcal{K}$. The compatibility with infinitary multiplication can be inferred analogously. Hence, H_p is a set of homomorphisms.

Since the mapping f is injective, for any two distinct $k, \ell \in \mathcal{K}$, there must be a prime ideal P containing exactly one k and ℓ. By definition, it holds that $h_P(k) \neq h_P(\ell)$. Thus, the set H_p is separating for $\mathcal{K}$. □

Since we have already shown that each semiring $\mathcal{K}$ for which there is a separating set of homomorphisms $h\colon \mathcal{K} \to \mathbb{B}$ must be multiplicatively idempotent and absorptive, proposition 5.21 illustrates that a separating set of homomorphisms $h\colon \mathcal{K} \to \mathbb{B}$ exists exactly for distributive lattices $\mathcal{K}$, provided that the infinitary operations are defined by the supremum or infimum of the finite subsums or subproducts. Based on the separating set H_p of homomorphisms, we can derive the following formulation of the homomorphism game HG_m^{inf} which applies to distributive lattices of arbitrary cardinality with appropriate infinitary operations.

Definition 5.22. In each play of $HG_m^{inf}(\pi_A, \pi_B)$, Spoiler first chooses $\pi \in \{\pi_A, \pi_B\}$ and some prime ideal $P \in I_p(\mathcal{K})$. Afterwards, Spoiler chooses some $a \in A$ or $b \in B$ and Duplicator has to respond with an element a or b in the other structure, which is repeated m times. Thus, each play results in a position $(\pi, P, a_1, \dots, a_m, b_1, \dots, b_m)$ and is won by Duplicator, if and only if, for each $L(\bar{x}) \in \mathrm{Lit}_m(\tau)$,

$$\begin{aligned}&\pi_B(L(\bar{b})) \in P \text{ implies } \pi_A(L(\bar{a})) \in P \text{ if } \pi = \pi_A \text{ and}\\ &\pi_A(L(\bar{a})) \in P \text{ implies } \pi_B(L(\bar{b})) \in P \text{ if } \pi = \pi_B.\end{aligned}$$

For each prime ideal $P \in I_p(\mathcal{K})$, we can construct formulae $\chi^{m,P}_{\pi_A,\bar{a}}(x_1, \dots, x_n)$ according to

$$\begin{aligned}\chi^{0,P}_{\pi_A,\bar{a}}(x_1, \dots, x_n) &:= \varphi^{=}_{\bar{a}}(\bar{x}) \wedge \bigwedge \{L(\bar{x}) \in \mathrm{Lit}_n(\tau) | \pi_A(L(\bar{a})) \notin P\} \text{ and}\\ \chi^{m+1,P}_{\pi_A,\bar{a}}(x_1, \dots, x_n) &:= \bigwedge_{a\in A} \exists x\, \chi^{m,P}_{\pi_A,\bar{a},a}(\bar{x}, x) \wedge \forall x \bigvee_{a\in A} \chi^{m,P}_{\pi_A,\bar{a},a}(\bar{x}, x),\end{aligned}$$

which characterize m-equivalence of $\mathcal{K}$-interpretations in distributive lattices as follows.

Theorem 5.23. Let $\mathcal{K}$ be a distributive lattice with infinitary operations defined via supremum and infimum of the finite operations. Given any two $\mathcal{K}$-interpretations π_A, π_B, elements $\bar{a} \in A^n$, $\bar{b} \in B^n$ and $m \in \mathbb{N}$, the following are equivalent:

(1) Duplicator wins $HG_m^{inf}(\pi_A, \bar{a}, \pi_B, \bar{b})$.

(2) For each $P \in I_P(\mathcal{K})$, it holds that $\{\pi_B[\![\chi^{m,P}_{\pi_A,\bar{a}}(\bar{b})]\!], \pi_A[\![\chi^{m,P}_{\pi_B,\bar{b}}(\bar{a})]\!]\} \cap P = \varnothing$.

(3) $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$

Chapter 6

Conclusion

In this thesis, we have examined the applicability of the classical Ehrenfeucht-Fraïssé game as a proof method for m-equivalence of $\mathcal{K}$-interpretations and discussed certain modifications of the game rules. As a first main result, we derived that the m-turn Ehrenfeucht-Fraïssé game does not capture m-equivalence under semiring semantics in any semiring which is not isomorphic to the Boolean semiring. As illustrated by the following $\mathbb{N}$- and $\mathcal{K}_4$-interpretations, this is due to the fact that semiring semantics admits counting of certain valuations in semirings which are not fully idempotent and, on the other hand, that the semiring elements are not accessible in the first-order formulae.

$(\mathbb{N}, +, \cdot, 0, 1)$

A	R	$\neg R$
a_1	1	0
a_2	1	0
a_3	2	0

$\not\equiv_1$

B	R	$\neg R$
b_1	1	0
b_2	2	0
b_3	2	0

Duplicator wins G_1

$(\{0, 1, 2, 3, 4\}, \max, \min, 0, 4)$

A	R	$\neg R$
a_1	1	0
a_2	2	0
a_3	4	0

$\equiv_1$

B	R	$\neg R$
b_1	1	0
b_2	3	0
b_3	4	0

Spoiler wins G_1

While full idempotence characterizes the class of semirings such that G_m is a sound proof method for m-equivalence, we have shown that the algebraic property is not necessary for elementary equivalence to be implied by Duplicator winning the game G. In order to account for counterexamples the soundness of G_m fails at, such as the $\mathbb{N}$-interpretations above, we also applied the m-turn bijection game and, equivalently, the m-turn counting game to $\mathcal{K}$-interpretations, which allow us to drop the requirement of full idempotence and infer m-equivalence based on a winning strategy for Duplicator in any semiring. However, the game rules aim at logics that enable counting to the full extent, which is appropriate for complex semirings such as $\mathbb{N}$ or $\mathbb{N}[X]$ but does not apply to a multitude of semirings, as arises from the $\mathbb{W}[\{x,y\}]$-interpretations below. Partially, we were able to account for different semirings by bounding the cardinality of the sets to be chosen in the counting game.

$$\begin{array}{c||c|c} A & R & \neg R \\ \hline a_1 & x+y & 0 \end{array} \quad \not\equiv_1 \quad \begin{array}{c||c|c} B & R & \neg R \\ \hline b_1 & x+y & 0 \\ \hline b_2 & x+y & 0 \end{array} \quad \equiv_1 \quad \begin{array}{c||c|c} C & R & \neg R \\ \hline c_1 & x+y & 0 \\ \hline c_2 & x+y & 0 \\ \hline c_3 & x+y & 0 \end{array}$$

We have shown that both the classical Ehrenfeucht-Fraïssé game and the bijection game on $\mathcal{K}$-interpretations are invariant under any injective mapping into another semiring, which illustrates that the game rules do not take into account the semiring operations in any way. Further, this arises from our observation that the notion of equivalence the game G_m on $\mathcal{K}$-interpretations characterizes can be expressed in terms of the induced classical τ^K-structures, which only depend on the cardinality of the underlying semiring but not on the operations. However, the varying algebraic properties of the semiring are crucial for the expressive power of first-order logic with semiring semantics and their incorporation into the game rules constitutes the main difficulty in search of a general game-theoretic characterization of m-equivalence between $\mathcal{K}$-interpretations.

Therefore, we restricted our analysis in section 5.2 to distributive lattices. We have seen that full idempotence as well as absorption enables the construction of sufficiently many homomorphisms into the Boolean semiring such that m-equivalence between two given $\mathcal{K}$-interpretations reduces to m-equivalence of $\mathbb{B}$-interpretations. Thus, we adapted the classical Ehrenfeucht-Fraïssé game to $\mathbb{B}$-interpretations which are not necessarily model-defining and finally derived the homomorphism game, which captures m-equivalence of $\mathcal{K}$-interpretations in any distributive lattice.

It remains open for future research to define further games which characterize m-equivalence for other, in particular, more general classes of semirings. Separating sets of homomorphisms into the Boolean semiring only exist for distributive lattices, which is why the homomorphism game cannot be directly extended to further semirings. However, depending of the $\mathcal{K}$-interpretations in question, it may not be necessary to separate each pair of semiring elements but only those that actually correspond to the valuation of a formula in the given $\mathcal{K}$-interpretations. Thus, a generalization of the game might be obtained by first excluding certain pairs of semiring elements and afterwards proceeding with the homomorphism game with a set of homomorphisms which separates the remaining pairs. Based on this idea, we proved m-equivalence of multiple $\mathcal{K}$-interpretations in chapter 3.1 in order to validate counterexamples for the completeness of G_m.

Leaving the natural definition of m-equivalence with respect to semiring semantics, it is still open whether there is a logic which does not contain all semiring elements or a set of generators, whose expressive power is captured by the game G_m on $\mathcal{K}$-interpretations. Two-sorted structures with varying relations and functions in the secondary structure, as introduced in section 4.2, may serve as a starting point for further analysis. Beyond that, we might also investigate the effect of dropping the requirement that quantifiers only range over elements of the primary universe. For instance, we could include a second sort of quantifiers ranging over the semiring elements and analyze the expressive power of the resulting logic.

Another follow-up question, concerning the applicability of the standard games, is which further semirings, besides $\mathbb{N}$ and $\mathbb{N}[X]$, are expressive enough such that m-equivalence is captured by the m-turn bijection game and whether we can characterize them algebraically.

Beyond that, we should also work on negative results. In chapter 3, we analyzed several semirings that differed significantly in the applicability of the standard Ehrenfeucht-Fraïssé games, which gives rise to the intuition that there might not be a feasible game which characterizes m-equivalence for each semiring $\mathcal{K}$. In order to verify this intuition, it remains to specify which games we consider feasible and to derive invariants from that, which would also have to apply to the notion of m-equivalence to be captured.

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